Studies on Non-Planar Computer Generated Holograms for 3D Display

by

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Declaration of Authorship

I, BOAZ JESSIE JACKIN, declare that this thesis titled, ‘A STUDY ON NON-PLANAR COMPUTER GENERATED HOLOGRAMS’ and the work presented in it are my own. I confirm that this work was done wholly or mainly while in candidature for a research degree at this University. It contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of the university or other institute of higher learning, except where due acknowledgment has been made in the text.

Signed:

Date:
“It is the study of light that shines light on my quest for knowledge”
Abstract

This research work is an investigation on diffraction theories and the availability of fast computation methods for non planar computer generated holograms. By non planar holograms we consider cylindrical and spherical surfaces for hologram computation. The theory for computer generated holography and has been extensively developed and fast computation methods are readily available but for holograms generated on plane surface only. Moreover lithographic printing and optical reconstruction are very easy if the hologram is made on a plane surface. Due to the above reasons computer generated holograms are usually generated for plane surfaces by considering the object to be parallel plane surface or set of parallel plane surfaces. This method has an important drawback that the object cannot be reconstructed from all sides for 360°, which is possible by using cylindrical or spherical hologram. Moreover With the availability of optic fibers and precise lithographic devices, the optical reconstruction of a cylindrical or spherical computer generated holograms becomes less difficult. Hence motivated by these facts, it is worthy to research on diffraction theories for cylindrical and spherical computer generated holography and develop fast computation methods that does not exist at present.

The investigation started with the development of wave propagation formula for cylindrical geometry and in spectral domain. The formula was derived as the boundary value solution to the Helmholtz equation. The wave spectrum which defines the decomposition of wavefield on cylindrical surface is defined. The propagation of the wave spectrum was defined by the transfer function which is a ratio of Hankel functions. Using these definitions a fast computation method for cylindrical computer generated holograms was developed which used fast Fourier transforms for its calculations. Using the method cylindrical holograms were computed, initially for cylindrical objects and then for arbitrary 3D object. The results were verified by comparing it with direct integration method results. The simulated reconstruction of the object from cylindrical holograms resembled well with the object. The optical reconstruction of cylindrical hologram was also successfully demonstrated.

Then a more complex situation was considered which is to generate a computer generated spherical hologram. Here again the system was considered as a boundary value problem and solutions were obtained for the Helmholtz wave equation by solving it using variable separable method. From the solution, the wave spectrum and transfer functions were defined. Since a sphere is a closed surface the spectral components take only integer values and does not extend to infinity, which is a fundamental difference. Fast computation is not directly achievable due to the uneven sampling in spherical
grids which destroy the shift invariance. However the shift invariance was restored by using the theory of band limited square integrable functions on a sphere and hence fast computation methods could be used. The developed computation method resembles the angular spectrum method very much, but differs by the fact that, the former uses spherical harmonic transform while the later uses Fourier transform for it evaluation. The simulated spherical hologram patterns were compared and verified with the ones generated by direct integration method. Form the generated spherical hologram the spherical object was reconstructed successfully. Three dimensional reconstructions were also demonstrated.

Both cylindrical and spherical holograms comes under the category of non-planar holograms. This research work is an entirely new attempt to extend the diffraction formulas and computation schemes in computer generated holography to non planar surfaces. The methods were developed as boundary value solutions to the scalar wave equation and tested successfully for simulated and optical reconstructions. Hence this work will add another dimension to computer generated holography and also provide more insights into spectral decomposition of wave fields in 360°.
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<td>VTK</td>
<td>Visualization Tool Kit</td>
</tr>
<tr>
<td>CPU</td>
<td>Central Processing Unit</td>
</tr>
<tr>
<td>GP-GPU</td>
<td>General Purpose Graphical Processing Unit</td>
</tr>
<tr>
<td>FFT</td>
<td>Fast Fourier Transform</td>
</tr>
<tr>
<td>DFT</td>
<td>Discrete Fourier Transform</td>
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<tr>
<td>SHT</td>
<td>Spherical Harmonic Transform</td>
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<td>AS</td>
<td>Angular Spectrum</td>
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## Symbols

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<td>°</td>
<td>Degree</td>
</tr>
<tr>
<td>cm</td>
<td>Centimeter</td>
</tr>
<tr>
<td>λ</td>
<td>Lamda</td>
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<td>θ</td>
<td>Theta</td>
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<td>ϕ</td>
<td>Phi</td>
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<tr>
<td>π</td>
<td>Pi</td>
</tr>
<tr>
<td>μm</td>
<td>Micrometer</td>
</tr>
<tr>
<td>i</td>
<td>Imaginary coefficient in complex number</td>
</tr>
<tr>
<td>E</td>
<td>Electric field vector</td>
</tr>
<tr>
<td>H</td>
<td>Magnetic field vector</td>
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<tr>
<td>$Y_{n}^{m}(\theta, \phi)$</td>
<td>Spherical harmonics</td>
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<tr>
<td>$P_{n}^{m}(x)$</td>
<td>Associated Legendre Polynomial</td>
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<td>$\bar{P}_{n}^{m}(x)$</td>
<td>Orthonormalized associated Legendre polynomials</td>
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<td>$H_{n}^{(1)}(x)$</td>
<td>Hankel functions of first kind</td>
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Dedicated to my loving wife Fanny and my dear parents
Chapter 1

Introduction

1.1 Introduction

The term holography is well known to the common man as, “depth perception from two dimensional surface or film”. It has become so popular because the image from a good hologram resembles the original object itself. But in scientific terms, holography cannot be restricted just to be described as a display technique with an extra dimension. Even the discovery of this technique did not originate from any optical recording or display related research. This technique was invented by Dennis Gabor in 1947 when he was working on to improve his electron microscope [1–3]. Holography is so popular as a display device because we have good coherent sources only in the visible spectrum. But the principles of holography suggests that, it is much more capable than that. In order to realise and appreciate all the potentials of this technique, it is necessary to understand the science behind it. Hence section 1.2 of this chapter introduces the technique of holography from a scientific perspective in more general terms. Then, the purpose of doing this work, and the motivation behind explained in section 1.3 and section 1.4. Section 1.5 explored the relationship between the proposed research and earlier existing methods. Section 1.6 presents the organisation of the thesis in accordance with the purpose mentioned.

1.2 Holography

Holography is the art of recording the complete information about an object and then reproducing it at a later time. Technically speaking, this is the recording of both amplitude and phase information contained within the wave emanating from an object.
Optically, hologram can be defined as “a diffraction screen which when suitably illuminated, diffracts light in a desired manner”. This diffracted light from the screen (hologram) can be made to resemble the light than would otherwise emanate from an object, when illuminated. Then if we look through the illuminated screen (hologram), we feel like viewing the original object itself. The diffraction pattern inscribed on the hologram is responsible for the whole process. The art of generating and inscribing that pattern on the screen (hologram) is called as holographic recording.

Holographic recording is very similar to photography. In photography, the light from the object (object beam) is allowed to fall on a recording material, which records only the intensity distribution of that object as shown in Figure 1.1. During replay only this intensity distribution across the object will be reconstructed on a 2 dimensional surface and hence no depth perception is possible. In holography, we do the same recording but with an additional beam called reference, provided both object beam and reference beam are coherent (i.e using a laser) as shown in Figure 1.2. This time the recording material again records the intensity distribution, but now it is the interference pattern due to the superposition of object wave and reference wave (Figure 1.2(a)). This recorded pattern is said to posses all the information (amplitude and phase) present in the wavefront that emanated from the object. Hence the term “holography” which is a compound of
the Greek words “holos = complete” and “graphein = to write”. In other words, the amplitude and phase information from the object are encoded as intensity modulations in the recording medium. This recorded pattern is called as hologram. When suitably illuminated with laser (Figure 1.2(b)), it is capable of reconstructing the original wave front that earlier emanated from the object. Hence any one looking into the hologram will have an illusion of looking into the original object itself.

Dennis Gabor’s solutions had to wait for 30 years until the invention of laser, for the first hologram to be made. The earliest and the popular method of recording the pattern on the hologram is by using highly coherent light sources (laser) and high resolution photosensitive recording materials [4]. The object was illuminated with laser, and the light that bounces back from the object reaches the recording medium. Another beam from the same laser directly reaches the recording medium. Both these beams interfere in the recording medium to produce the hologram. Different recording setups were proposed each having their own significances. Even holograms that could be reconstructed with an incoherent light like sun light were made using this procedure. The perception of a good quality hologram is really breathtaking. This method of recording and reconstruction is generally termed as conventional holography or optical holography.

Optical holography demands the presence of the real object (for which the hologram is to be made) during the recording process. So what if we need to produce the hologram of an object that never existed? The solution came from Lohman and Paris [5] in the form of computer generated holograms. In this process, the whole optical holographic recording setup (including the object) is simulated to generate the pattern that the real object would otherwise produce. Now the generated pattern is transferred to a transparency film and reconstructed using a laser. This was very attractive, because we have virtually created an object that never existed. This also provided other conveniences like not demanding a costly optical recording setup or vibration isolation arrangement. Scalar diffraction theories [6] were used to simulate all the process as (diffraction, wave propagation and interference) involved in the optical recording setup. This method of recording and reconstruction is popularly known as Computer generated holography or Digital holography.

In optical holography, the fringe pattern is recorded in films coated with photosensitive materials like, dichromated gelatin, silver halide etc. These materials require a dark room for recording process and then wet chemical processing for developing the film. This is a very tedious process which limits the capabilities of holography. More over the recorded pattern is static on the film and hence cannot be subjected to image processing or digital signal processing on a computer for more information. Hence with the development of high resolution electro-optic recording devices like CCD, these
conventional recording plates were replaced. So instead of reconstructing virtually or on a screen with laser, the reconstruction was done on a computer. This opened up more possibilities and revealed more interesting details about the recorded object. This method of recording and reconstructing is also termed as Digital Holography [7]. It has a lot of potential applications with microscopy and particle velocimetry being the prominent ones. Devicing a new and more efficient method to make such digital holograms is the aim of the proposed research work. Such an attempt is reported in this thesis with results.

1.3 Problem Statement

Holograms can record and reproduce all the three dimensional informations like motion parallax, accommodation, occlusion, convergence and so on. Hence it is possible to perceive a three dimensional object very close to reality, when looking into a hologram, from any direction. But the potential of the holography is restricted by the geometrical shape of the hologram. Usually holograms were made on flat surfaces, which has a limited viewing angle. It is not possible to view or record any information about the back side of the object. This constraint has been overcome by making holograms on cylindrical surfaces. A cylindrical hologram has a “look around property” and hence can be observed from any direction. Such holograms were developed optically from a real existing object and the optical reconstructions could reconstruct from all directions [8, 9]. It will be more interesting if such a cylindrical hologram can be generated in computer ie. for an object that never existed. Hence this research is intended to make a cylindrical computer generated hologram using a new method.

Even though the cylindrical hologram can reconstruct the object in 360° in horizontal direction, information from the top and bottom sides of the object is still lost. To overcome this problem the holographic surface should be considered as a spherical one. The spherical hologram is closed from all directions and could reconstruct the object for 360° in both horizontal and vertical direction. So during reconstruction it is possible to observe the object from top and bottom also. Due to these interesting properties it was decided to do some research on computation methods for generating spherical holograms on the computer.

1.4 Motivation and challenges

Cylindrical geometry has been very efficient in 3D display and also volumetric imaging such as in Magnetic Resonance Imaging and Computed Tomography. Hence using
Introduction

cylindrical geometry for computer generated holography or digital holography has more potentials compared to the conventional planar counterpart. Owing to these appealing facts more interest has been generated in computer generated cylindrical holography in the recent years [10–13]. However the two major constraints in achieving this are the optical setup and the numerical procedures which are still in its infancy. The constraint due to optical setup can be overcome by the recent availability of variety of optic fibers. Due to these facts it is necessary to do more development to numerical procedures. This motivates us to do this research which is an attempt to develop a fast computation numerical method for computer generated cylindrical holography. The important challenge in this research is to find a solution to wave propagation from cylindrical surfaces that enables a fast computation process. The second one is the challenge to print the hologram and reconstruct it optically.

As explained earlier, spherical holograms have clear advantages over the cylindrical and planar ones. However there are only a very few papers [14, 15] reporting holograms on spherical surfaces or hemispherical surfaces. This is due to the fact that illuminating a spherical surface completely using a coherent source is extremely difficult. The other problems arise from printing the hologram and mounting it for optical reconstruction. However with the advancement in optic fibers and high precision lithographic machines this is no more a difficult task. Motivated by these facts, it was planned to do some research to improve the techniques and procedures that exist for spherical computer generated holography. Accordingly, the aim was to develop a new numerical procedure and programming scheme in order to achieve faster and efficient generation of the spherical hologram on the computer. Altogether since both cylindrical and spherical surfaces are considered, this research can be generalized as non-planar CGH.

1.5 Relationship to existing approaches

The recent developments in technology and the possibility of producing non-planar display and recording devices, has made people focus on non-planar geometries for holography. As a result, papers describing cylindrical and spherical computer generated holography started to appear in the recent five years. Y.Sakamoto et al [10] used the angular spectrum of plane waves method to generate a cylindrical hologram of a plane object. They employed the shift invariance in rotation between a planar and cylindrical surface and hence could use FFT. Then they improved on their method to generate the hologram of a volume object by slicing it into planar segments [12]. This took 2.76 hrs to calculate the hologram of a 13×13×13 mm object. Yamaguchi et al [13] used the Fresnel transform and segmentation approach to generate cylindrical holograms. Since
they did not use FFT, the computation time was 81 hrs on a parallel computing machine for an object of size $15 \times 15 \times 15$ mm. They also developed a computer generated cylindrical rainbow hologram using the same method [16]. The calculation time for the rainbow hologram was 45 min on a single computer, but sacrifices vertical parallax. Sando et.al [11] generated cylindrical holograms by defining propagation in spatial domain using convolution. In this method the shift invariance was preserved by choosing the object also to be a concentric cylindrical surface with the hologram. Hence FFT could be used and they had used three FFT loops for simulating wave propagation. Spherical surfaces were also considered in the past for computer generated holography. Fast computation of holograms on half spherical surface was reported by J.Rosen [15] where he used angular spectrum method for calculation. Fast computation solutions for spherical computer generated hologram employing PSF(convolution method) was proposed by Tachiki et al. [14]. Here the object and hologram both were concentric spherical surfaces and hence shift invariance was preserved which enabled them to use FFT for calculations. It is clear from the above discussion that so far no one has computed a cylindrical hologram or spherical hologram by considering wave propagation in spectral domain. This thesis reports such an approach to develop spectral wave propagation solutions to cylindrical and spherical surfaces. The expected advantages are i) better understanding of spectral decomposition of wave field from these surfaces, ii) faster computation speed and iii) more accuracy.

1.6 Organisation of Thesis

This thesis is organized into 6 chapters to make the reader understand the basics of the technique, the procedure followed in this work and the usefulness of the results. We start with explaining the basics of holographic recording and reconstruction with the necessary mathematical equations in Chapter 1. Then the various methods available for holography and computer generated holography are presented with their merits and demerits in Chapter 2. This will give an idea on the improvements needed to holography and also appreciate the need for the work reported in this thesis. The derivation of the spectral propagation formula for cylindrical surfaces from the fundamental wave equation is presented in Chapter 3. The fast computation procedure and the reconstruction results for cylindrical computer generated holography is discussed in Chapter 4. Chapter 5 presents the derivation of spectral wave propagation formula for spherical surfaces. The fast computation of spherical hologram and corresponding reconstruction results using the proposed formula are presented in Chapter 6. Chapter 7 summarises the research with concluding remarks.
1.7 Conclusion

A very brief introduction to the concept of holography has been presented in very general terms. From this general introduction, the motivation and the purpose of doing this work has been explained. Accordingly, the purpose is to develop a new fast computation method for computer generated cylindrical and spherical holograms based on spectral wave propagation formula. The expected advantages of this research work is also presented.
Chapter 2

Background

2.1 Introduction

The art of holography has undergone several changes since its invention. The techniques and principles of holography have been modified and improved through years to achieve better efficiency and quality. As explained earlier the aim of this work is also the same, which is to devise new calculation methods for non-planar holography. Among the vast number of methods available, each has its own merits, and drawbacks. None of them are superior than all others in every aspects. So a suitable method should be chosen based on the problem in hand. For this, a thorough analysis of all the evolved methods, their properties including difficulties, is necessary. Hence this chapter will present in detail, the basic theory of holography and the available methods and techniques for holographic recording and reconstruction. The discussion in this chapter will also make clear the need for non-planar shaped hologram, and explain its advantages and disadvantages.

2.2 Optical Holography

The method of holography in which the recording and reconstruction is done using laser light on a holographic plate is called as optical holography. This was the earliest and popular method for making holograms which demands the presence of real object and highly stable vibration free recording environments. The reconstruction is done either with laser or white light. The best quality holograms that very closely resemble the object were made using this procedure. No digital electronic devices were used for recording or reconstruction. Hence the name optical holography.
Later on with the development in opto-electronic devices digital electronics began to take part in holography. This gave rise to an interesting field of research namely, Computer generated holography. Holograms of non-existing objects are made using this method. Real time dynamic imaging and reconstruction were also made possible using these methods, which was a major drawback in optical holography.

Eventhough the method was digitised, the theory and methods of optical holography still apply for digital holography. Computer generated holography and digital holography were built on the principles of optical holography. In other words, optical holography is the forefather of digital holography. The work reported in this thesis is also an attempt to digitise such an optical holographic techniques that include cylindrical and spherical holography. Therefore a discussion on the fundamentals of optical holography and its various methods will throw more light on the foundations of this research work. The following sections explains the basics and various methods available for optical holography with their merits and demerits.

### 2.2.1 Principle

Light is electromagnetic in nature and hence the theory of holography entirely revolves around the equations of electromagnetic wave propagation and their solutions [6, 17]. A typical holographic recording and reconstruction setup considered for explaining the theoretical foundations of holography is shown in Figure 2.1.

The principle of optical holographic recording is shown in Figure 2.1(a) which consists of the light source, object (to be recorded) and a recording device, e.g. a photographic plate. Light with sufficient coherence length is split into two partial waves using a beam splitter. The first wave illuminates the object and is called as the object wave. It is scattered at the object surface and reflected to the recording medium. The second wave, named the reference wave, illuminates the light sensitive medium directly. Since they are coherent, both waves interfere to create a standing wave pattern. The interference pattern is recorded by chemical development of the photographic plate. The recorded interference pattern is known as the hologram. The hologram has recorded all the information that came from the object, i.e, both phase and amplitude.

To get back the recorded information, the hologram is illuminated with the same reference beam alone. In other words, the original object wave is reconstructed by illuminating the hologram with the reference wave as shown in Figure 2.1(b). An observer viewing through the hologram sees a virtual image of the object, which resembles the original object itself. There is also a real image and other wavefronts reconstructed,
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(a) Recording

(b) Reconstruction

Figure 2.1: Optical Holography
which will be explained below. The reconstructed image exhibits all effects of perspective and depth of focus. The above mentioned recording and reconstruction process can be explained in the language of Mathematics as follows.

The complex amplitude of the object wave is described by

\[ U(x, y) = A_0(x, y)e^{i\phi(x, y)} \]  

(2.1)

with real amplitude \( A_0(x, y) \) and phase \( \phi(x, y) \).

\[ R_r(x, y) = A_r(x, y)e^{i\psi(x, y)} \]  

(2.2)

is the complex amplitude of the reference wave with real amplitude \( A_r(x, y) \) and phase \( \psi(x, y) \). Both the waves interfere at the surface of the recording medium resulting in an intensity distribution (fringe pattern) across the medium. This intensity distribution can be calculated as follows.

\[ I(x, y) = |U(x, y) + R_r(x, y)|^2 \]  

(2.3)

\[ = |A_0(x, y)|^2 + |A_r(x, y)|^2 + 2A_0(x, y)A_r(x, y)\cos(\psi(x, y) - \phi(x, y)) \]  

(2.4)

where the last term equals \( UR_r^* + U^*R_r \) and includes both the amplitude and phase of the object wave front, i.e., \( A_0(x, y) \) and \( \phi(x, y) \).

The transmission function of optical recording devices including photographic film is sensitive to intensity. We will assume that the sensitivity is linear in intensity. The reference \( A_r(x, y) \) will be assumed to be constant, and equal to \( A \), which is a plane wave incident perpendicular to the hologram. The transmission function of such a device can be written as

\[ t(x, y) = t_0 + \beta \tau \left[ |A_0(x, y)|^2 + |A_r(x, y)|^2 + UR_r^* + U^*R_r \right] \]  

(2.5)

where \( \beta \) and \( t_0 \) are constants. The constant \( \beta \) is the slope of the amplitude transmittance versus exposure characteristic of the light sensitive material. For photographic emulsions \( \beta \) is negative. \( \tau \) is the exposure time and \( t_0 \) is the amplitude transmission of the unexposed plate. \( t(x, y) \) represents the stored information and is known as the hologram function. In Digital holography where CCD’s are used as recording medium the term \( t_0 \) can be neglected. Now suppose that the generated hologram is illuminated by another reference wave \( R(x, y) \) as shown in Figure 2.1(b). The wave emanating from the hologram can be written as

\[ u(x, y) = (t_0 + \beta \tau \left[ |A_0(x, y)|^2 + |A_r(x, y)|^2 + UR_r^* + U^*R_r \right]) R(x, y) \]  

(2.6)
\[ R_t = U_1 + U_2 + U_3 + U_4 \]  
\[ \text{where,} \]
\[ U_1 = (t_0 + \beta \tau |A_r(x, y)|^2)R(x, y) \]  
\[ U_2 = \beta \tau |A_0(x, y)|^2 R(x, y) \]  
\[ U_3 = \beta \tau R_r(x, y)^*U(x, y)R(x, y) \]  
\[ U_4 = \beta \tau R_r(x, y)U(x, y)^*R(x, y) \]

Suppose that \( R_r \) and \( R \) are the same and are constant, as in a plane wave case perpendicular to the direction of propagation. Then, \( U_3 \) is proportional to \( U \), and \( U_4 \) is proportional to \( U^* \).

The first term \( U_1 \) refers to the intensity reduction of the reconstruction wave by the factor \( t_0 + \beta \tau |A_r(x, y)|^2 \) during reconstruction. The second term is small assuming that we choose \( A_0(x, y) < A_r(x, y) \) during recording. This term is distinguished from the first term by its spatial variation \( |A_0(x, y)|^2 \). The \( |A_0(x, y)|^2 \) term contains low spatial frequencies which have small diffraction angles and create a so-called halo around the reconstruction wave. The size of the halo is given by the angular dimension of the object. These first two terms form the zeroth diffraction order in equation. The third term \( U_3 \) in Equation (2.6) denotes the object wave \( U(x, y) \) multiplied with the constant factor \( \beta \tau R_r^2 \). An observer who registers this wave in his eye therefore sees the virtual image of the (not present) object. The third term is the most important and represents the first diffraction order. The wave travels divergent from the hologram thus creating a \textit{virtual image} at the position of the original object. It is a virtual image because the wave is not converging to form a real image. This image cannot be captured on a screen. The intensity (square of amplitude) of the image does not depend on the sign of \( \beta \). Therefore it is unimportant whether the hologram is processed “positive” or “negative”.

The fourth term \( U_4 \) is essentially the complex conjugate of the object wave \( U^* \) except for a multiplicative term. This represents the \(-1^{st}\) diffraction order. Since it is complex conjugated wave, the phase changes its sign with respect to \( U(x, y) \). As a consequence the wave \( U^*(x, y) \) travels convergent and forms a real image. The conjugated real image \( U_4 \) is usually located at the opposite side of the hologram with respect to \( U_3 \). \( U_3 \) and \( U_4 \) are called \textit{twin images} or also represented as virtual image and real image, respectively.

All these reconstructed wavefronts are represented in Figure 2.1(b). However, which image is virtual and which is real actually depend on the properties of the reference waves used during recording and reconstruction. These issues are further discussed in the next section.
The virtual image appears at the position of the original object itself, if the hologram is reconstructed with the same parameters like those used in the recording process. However, if one changes the wavelength or the coordinates of the reconstruction wave source point with respect to the coordinates of the reference wave source point used in the recording process, the position of the reconstructed image moves. The coordinate shift is different for all points, thus the shape of the reconstructed object is distorted. The image magnification can be influenced by the reconstruction parameters, too.

The Equations (2.12 to 2.17) are called imaging equations that relate the coordinates of an object point $O$ with that of the corresponding point in the reconstructed image. Only the final equations are mentioned here. The detailed derivations are given by Hariharan [4] and Kreis [18].

**Figure 2.2:** Formation of image point object by hologram
The coordinate system is shown in Figure 2.2. \((x_o, y_o, z_o)\) are the coordinates of the object point \(O\), \((x_r, y_r, z_r)\) are the coordinates of the source point of the reference wave used for hologram recording \(R\) and \((x_p, y_p, z_p)\) are the coordinates of the source point of the reconstruction wave \(P\). \(\mu = \lambda_2/\lambda_1\) denotes the ratio between the recording wavelength \(\lambda_1\) and the reconstruction wavelength \(\lambda_2\). The coordinates of that point in the reconstructed virtual image, which corresponds to the object point \(O\), are:

\[
x_1 = \frac{x_p z_o z_r + \mu x_o z_p z_r - \mu x_r z_p z_o}{z_o z_r + \mu z_p z_r - \mu z_p z_o} \tag{2.12}
\]

\[
y_1 = \frac{y_p z_o z_r + \mu y_o z_p z_r - \mu y_r z_p z_o}{z_o z_r + \mu z_p z_r - \mu z_p z_o} \tag{2.13}
\]

\[
z_1 = \frac{z_p z_o z_r}{z_o z_r + \mu z_p z_r - \mu z_p z_o} \tag{2.14}
\]

The coordinates of that point in the reconstructed real image, which corresponds to the object point \(O\) are:

\[
x_2 = \frac{x_p z_o z_r - \mu x_o z_p z_r + \mu x_r z_p z_o}{z_o z_r - \mu z_p z_r + \mu z_p z_o} \tag{2.15}
\]

\[
y_2 = \frac{y_p z_o z_r - \mu y_o z_p z_r + \mu y_r z_p z_o}{z_o z_r - \mu z_p z_r + \mu z_p z_o} \tag{2.16}
\]

\[
z_2 = \frac{z_p z_o z_r}{z_o z_r - \mu z_p z_r + \mu z_p z_o} \tag{2.17}
\]

An extended object can be considered to be made up of a number of point objects. The coordinates of all the surface points are described by the above mentioned equations. The lateral magnification of the entire virtual image is described as

\[
M_{\text{lat},1} = \frac{dx_1}{dx_o} \left[ 1 + z_o \left( \frac{1}{\mu z_p} - \frac{1}{z_r} \right) \right]^{-1} \tag{2.18}
\]

The lateral magnification of the real image is

\[
M_{\text{lat},2} = \frac{dx_2}{dx_o} \left[ 1 - z_o \left( \frac{1}{\mu z_p} + \frac{1}{z_r} \right) \right]^{-1} \tag{2.19}
\]

The longitudinal magnification of the virtual image is given by

\[
M_{\text{long},1} = \frac{dz_1}{dz_o} = \frac{1}{\mu} M_{\text{lat},1}^2 \tag{2.20}
\]

The longitudinal magnification of the real image is

\[
M_{\text{long},2} = \frac{dz_2}{dz_o} = \frac{1}{\mu} M_{\text{lat},2}^2 \tag{2.21}
\]
Apart from all these, there is a very important difference between the real and virtual image. Since the real image is formed by the conjugate object wave $U^*(x,y)$, it has the curious property that its depth is inverted. Corresponding points of the virtual image (which coincides with the original object points) and of the real image are located at equal distances from the hologram plane, but at opposite sides of it. The background and foreground of the real image are therefore exchanged. The real image appears inverted. This image is called “pseudoscopic” contrary to the normal image which is called “orthoscopic”.

2.2.2 Methods in Optical Holography

As mentioned earlier there are a lot of methods available for recording and reconstructing holograms optically. Each method has its own merits and demerits. This section discusses briefly each method and its significances. For a detailed description the reader may refer to Ackermann and Eichler [19] and Hariharan [4].

2.2.2.1 Inline hologram

For this type of hologram the object is a plane transparency containing small opaque details on a clear background. The object is illuminated by a collimated beam of monochromatic light along an axis normal to the holographic plate. The light incident on the holographic plate then contains two components. The first is the directly transmitted wave, which is a plane wave whose amplitude and phase do not vary across the photographic plate. The second is a weak scattered wave which emanates from the object. Both these waves superimpose on the photographic plate giving rise to fringe pattern which is the hologram. A detailed mathematical derivation for the fringe pattern formed is given by Hariharan [4]. During reconstruction the hologram is illuminated with the same a plane reference wave and is viewed from the other side. A virtual object is formed at the original object position and additionally a real image point appears at the same distance in front of the hologram. This was the earliest method and was developed by Gabor [1]. It was named after him as Gabor’s inline holography.

This method has certain demerits. During observation the two images lying on the same axis interfere which leads to image disturbances. Moreover, the observer looks directly into the reconstruction wave, which is not always safe. But this method has its own advantages. A single laser beam is used for the recording which constitutes both the object and reference beam without splitting the beam. This technique is also called as “single beam holography”. The fringe density is very low compared to the
other methods where there is an angle between the object and reference beams. This significantly reduces the computation load and is of great help to digital holographers.

2.2.2.2 Off-Axis Hologram

In off-axis holography the reference beam is derived from the same source using a beam splitter. Then by tilting the reference wave (or shifting the object) the three diffraction orders, namely the image, the conjugated image, and the illumination wave, are spatially separated [20, 21]. Hence the unwanted overlapping of the real and virtual image suffered by the inline recording method is avoided. This also has the advantage that, holograms of opaque objects can be produces since the reference wave is not obstructed by the object.

This is the most popular and most used method for recording holograms optically. But on the contrary this is the least used method by digital holographers. This is due to the fact that, the increase in angle between the reference and object increases the fringe density as well. Hence the number of samples should be increased in order to completely record all information which in turn affects the computation speed. Hence digital holographers always prefer Gabor’s inline recording setup. It is also worth noting that inline recording setup is used for the work reported in this thesis.

2.2.2.3 Fourier Hologram (Lensless)

If the object and the reference are within the same plane parallel to the hologram, then the so called “Fourier holograms” are generated. It is also necessary that the reference should be a point source and the object is illuminated with a plane wave. Then a hologram which is similar and has the same properties as that of a Fourier hologram is generated. Since this is a Fourier hologram generated without a lens it is called as lensless Fourier hologram [22].

The special property of this hologram is that, like in all thin holograms two images appear during reconstruction but both are virtual now. The regular image is in the position of the original object, while the conjugated one appears in the same plane parallel to the hologram. The point light source that represents the reference will be the center of point symmetry for the two images. The other properties of these holograms are same as a Fourier hologram and are discussed in the next section.
2.2.2.4 Fourier Hologram

The object is a plane and is placed in the first focal plane of the lens. The reference wave emerges from a point light source in the same plane. The holographic plate is placed in the back focal plane of the lens during recording [23]. The reconstruction is done by illuminating the hologram with and axially parallel plane wave. The hologram is again placed in the first focal plane of a similar second lens. The primary and the conjugated images appear in the second focal plane symmetric to the optical axis. The undiffracted reference wave forms an axial light spot representing the zeroth diffraction order. It can be shown that the reconstructed image remains stationary when the hologram is shifted in its plane.

Fourier holograms have the useful property that the reconstructed image does not move when the hologram is translated in its own plane. This is because a shift of a function in the spatial domain only results in its Fourier transform being multiplied by a phase factor which has no effect on the intensity distribution. This setup is most liked by digital holographers because the simulation is very easy which only requires an FFT calculation. The Fourier holograms are limited only to plane objects and 3D perception is not possible.

2.2.2.5 Fraunhofer Hologram

As explained in the previous section, Fourier holograms are formed by the superposition of spherical waves whose centers have the same distance from the holographic layer. If the layer is moved far away, the centers depart and in the limit plane waves are created. This kind of holograms are called “Fraunhofer holograms”.

A hologram of this type is especially used for the measurement and investigation of aerosols [24, 25]. The object has to be so small that a diffraction pattern will appear in the far field. The condition for the distance between the object and hologram is $z \ll r^2/\lambda$, where $z$ is the distance and $r$ is the radius of the object.

2.2.2.6 Image plane Hologram

It has a lot of advantages to record the real image of an object instead of the object itself. For image-plane holograms the object is imaged into the plane of a hologram by a large lens. During reconstruction, the real image of extended objects is partly in front of and behind the hologram.
Due to the hologram plane being in the middle of the image the differences in path lengths are smaller than those in other techniques. Hence minimal demands are made regarding the coherence of the light source. If the depth of the object is small even white light sources can be used. Another advantage is that image-plane holograms are relatively bright and brilliant, though the observation angle is limited by the lens aperture.

2.2.2.7 Rainbow Hologram

Rainbow holograms can be reconstructed in transmission using white light [26, 27]. Depending on the viewing direction the reconstructed image appears in different colors, exhibiting the whole light spectrum. The technique for the recording of rainbow holograms consists of two steps. In the first step an off-axis hologram is created in the usual manner. In the second step, a photosensitive layer is positioned inside the real image and a second hologram $H_2$ is created. By this process the information of the pseudoscopic image is recorded.

To reconstruct the images of rainbow holograms, they are rotated by $180^0$ to create an orthoscopic image from the pseudoscopic one. The image is reconstructed using monochromatic light. The observer looks through a horizontal slit which is the image of the aperture that was used. A high intensity is achieved since the diffracted light is concentrated on the slit. The viewing angle is limited and the three-dimensional impression exists only in the horizontal direction.

When using the white light for reconstruction, the image of the horizontal slit appears under a different diffraction angle. For each spectral color there exists a different viewing slit. If the observer moves the head in the vertical direction he or she will see the image successively in red, orange, yellow, green and blue, i.e., in the spectral colors of the rainbow. Hence this method has the name rainbow holography.

2.2.2.8 Double sided Hologram

Usually only the information of the front side of a three-dimensional object can be recorded on a plane hologram. With double-sided holograms the hologram can be viewed from two sides and the reconstructed image shows front and backside of the object.

The production of a double-sided hologram starts with the recording of a transmission master hologram $H_1$ of side (1) of the object. After that a second hologram $H_2$ of the wavefront from the other side (2) of the object recorded. This one is not developed at first and a latent reflection hologram is created. Starting from this hologram $H_2$ the
third step consists in creating a double sided hologram. In doing so a pseudoscopic real image of side (1) of the object is generated from the master hologram by inverting the direction of the reference wave. On the second hologram H2 a second exposure is made and the wavefront from the master is recorded. The direction of the reference wave is different from the first exposure. For the reconstruction the illumination wave has to be inverted again since the image of side (1) was pseudoscopic. Two independent reflection holograms are obtained which display both sides of the object by a virtual and a real image.

2.2.2.9 Reflection Hologram

Until now thin holograms were discussed where the object and reference wave impinges from the same side on the photographic layer. In the case of reflection holograms, the reference wave and the object wave impinges from the opposite sides of the photographic layer. Later the reconstruction wave impinges from the observer’s side onto the hologram during reconstruction [28–30].
The optical setup for recording a reflection hologram is shown in Figure 2.3(a). The holographic layer is positioned in between the light source and the object. This results in the interference planes being almost parallel to the light sensitive layer. The distance of the grating planes when using a He-Ne laser is $\lambda/2 \approx 0.3 \, \mu m$. So, for 20 grating planes to fit into the recording material, it has to be of almost 6 $\mu m$ in thickness. Hence the system behaves like thick grating.

The common diffraction theory has to be modified for the reflection holograms because thick gratings exhibit a totally different behavior. During reconstruction the illumination wave which is ideally identical to the reference wave is reflected at the grating planes. The virtual image of the object appears in the reflected light, see Figure 2.3(b). If white light is used for illumination only the wavelength used for the recording is reflected due
Background

2.2.2.10 Cylindrical Hologram

A drawback experienced by all the holographic methods discussed so far is the limited angle over which they can be viewed. This is because they were all made only on plane surfaces. Making a hologram in a cylindrical shape can solve this. The cylindrical holography is supposed to make the complete geometry of the object viewable. It can be recorded by using a cylindrical film surrounding the object. Figure 2.4 shows the setup for a single-beam transmission hologram, proposed by Jeong [8]. The object is placed at the center of a glass cylinder which has a strip of photographic film taped to its inner surface with the emulsion side facing inwards, and the expanded laser beam is incident on the object from above. The central portion of the expanded laser beam illuminates the object gets scattered and reaches the cylindrical hologram surface, which constitute the object beam. While the outer portions, which fall directly on the film, constitute the reference beam. The object and reference beams interfere in the cylindrical hologram surface to generate the hologram.

![Cylindrical Hologram - Recording setup](image)

To view the reconstructed image, the processed film is replaced in its original position and illuminated with the same laser beam. When illuminating, all perspectives of the object are displayed when walking around the hologram. Multiplex holograms are also often reconstructed using the 360° geometry. With an illumination from above,
all recordings of the multiplex hologram can be reconstructed simultaneously and the recorded changing images can be observed when walking around the complete circle. Hence this method is also called as 360° holography. But the optical recording setup imposes serious difficulties during recording. The geometric shape of the object may prevent some portion of the object being illuminated. Aligning the optical setup along the vertical direction in the optical table and mounting the hologram and object is also has serious problems. However, generating the hologram using computer can solve all these problem. Accordingly, generating a cylindrical hologram on a computer is the aim of the research work reported in this thesis.

Various holographic recording setups and the merits and demerits of using each one has been discussed. It is clear that none of the holographic setups are not able to reconstruct an object with the “look around property”, except the cylindrical holography. But as explained earlier it faces serious difficulties with the optical recording and reconstruction setups. The difficulties can be over come if the object is modeled on the computer and the hologram can be generated in the computer itself. Hence it was intended to do some research on generation of cylindrical holograms on computer and devise a more efficient computation method.

2.3 Computer Generated Holography

Technologies like photography, signal processing, terrestrial television broadcasting etc, which were born as analog ones, are all digitized today. This is mainly due to the efficiency and convenience that these digital systems offer when compared to their analog counterparts. The other reason is the revolution in electronic and computer research sectors, which kept pouring solutions and advanced instrumentation to any kind of problems in the digitization process. Holography was no exception to this trend and holographists also tried to digitize conventional optical holography. The first successful attempt was reported by Lohman and Paris [5]. Thus was born digital holography and it had many advantages and some disadvantages over its conventional (optical) counter part. The research work reported in this thesis is also an attempt to digitize a conventional optical holographic method called as cylindrical holography. In order to appreciate the usefulness of the work reported in this thesis, it is worth discussing the basics of digital holography. Hence this section explains the basic principles of digital holography and its various implementation methods with their advantages and disadvantages.

Holographic recording process is also possible without using any optics. It is usually done by simulating the optical holographic recording setup on a computer by modelling
the object to be recorded is modeled in the the computer. Then the wave propagation from object to the hologram plane is simulated. The same simulation is done for the reference, and the interference between the object and reference is calculated at the hologram plane. Hence the required hologram is generated. This generated hologram is in digital format and can be transferred to a photographic film using image setter or other methods. Then the photographic film can be reconstructed using optical methods. Thus hologram of an object that does not exist or cannot be optically recorded, can be produced using this method. This method is usually used to make display holograms and known by the term computer generated holography.

However the hologram can also be recorded optically using a CCD instead of holographic plate. Thus the recorded hologram is in digital format on the computer. Later the hologram is reconstructed on the computer screen or other interactive displays. For this the wave propagation is simulated from hologram plane to object plane. The object that was recorded can be viewed in the computer screen with all the 3D and depth information. This type of holographic recording and reconstructions in termed as Digital Holography.

Simulating wave propagation from object to hologram plane or vice versa, occurs in both the process and is the most important step. This is basically a signal processing problem. The research work reported in this thesis basically deals with display holography. Hence the following sections will discuss only problems related to display holography.

Two fundamental signal processing problems in holographic display are referred to as forward and reverse problems. The forward problem is the computation of the light field distribution which arises over the entire 3-D space from a given 3-D scene or object. In traditional holography, this light field would have been optically created and recorded by interferometric and other techniques, but in digital holographic systems the associated field must be computed. This is considerably more difficult problem because the 3-D scene consists of nonplanar surfaces. In other words simulation of wave propagation is the heart of computer generated holography.

Once the desired field is computed, physical devices will be used to create it at the display end. The field generated by these devices will propagate in space and reach the viewer, creating the perception of the original 3D-scene. These devices impose many constraints on the 3-D light distributions they can generate, as a consequence of their particular characteristic and limitations. Therefore, given a physical device, such as a specific SLM, finding driving signals to get the best approximation to the desired time varying 3-D light field is a challenging inverse problem. A precise definition of this, so called synthesis problem and some proposed solutions can be found in the literature [31–34].
Computation of propagating electromagnetic field depends on the foundations of diffraction theory [6, 35, 36]. Approaches in solving diffraction problems can be investigated under four categories. From rather simple to more complicated categories, these categories are ray optics, wave optics, electromagnetic optics and quantum optics. Ray optics describes the propagation of light by using geometrical rules and rays [37]. In wave optics, the propagation of light is described by a scalar wave function which is a solution of the wave equation. The work reported in this thesis also uses the wave optics for simulating wave propagation of light. Hence the theory of wave optics is presented in detail in section 2.3.1 and the various wave optic techniques and corresponding fast algorithms are reviewed in section ???. Based on the computation models many methods have been proposed for Computer generated holography which are explained in section 2.3.2.

Other signal processing approaches have also been extensively employed in problems related to wave optics. However the present state-of-the-art does not seem to be sufficient for solving some of the problems arising in real-time holographic, 3-D display. In order to facilitate further developments, several signal processing tools which has the potential of advancing the state-of-the-art has been discussed in section ??.

Another problem of fundamental nature is the discretization of signals associated with propagating optical waves. At the acquisition stage, CCD or CMOS arrays capture holographic patterns and convert them into digital signals [18, 38, 39]. While sampling and quantization is an extensively studied and mature field in the general sense, direct application of the general results will not be efficient in most diffraction related problems. Instead, systematic approaches which take the specific properties of the underlying signals into consideration and merge them with modern digital signal processing methods are highly desirable. The literature dealing with discretization and quantization issues in diffraction and holography are reviewed in section 2.3.3.

### 2.3.1 Electromagnetic Wave Propagation

Light is electromagnetic in nature and electromagnetic field any where in space is well defined by the Maxwell’s equations. The propagation of electromagnetic field is defined by the wave equation. Analytic solution to wave equation describes the wavefield due to a propagating wave front anywhere in space. But in digital holography, the object has arbitrary shape and size and hence analytic solutions to the wave equation is not possible. So numerical solution to the wave equation is sought to calculate the wavefield in the hologram plane or reconstruction plane. Wave equation is a vectorial differential equation and numerically solving it is very time consuming. Moreover sampling errors
and discretization errors creep in when the distance of propagation increases, affecting the results very badly. To overcome these issues, approximations have been induced into the equation based on the problem in hand. The approximated equations are integral equations derived from the Helmholtz differential equation using a suitable Greens function. These integral equations are scalar in nature and hence are also called as scalar diffraction formulas. These approximated solutions make calculation much easier and faster, but at the same time give satisfying results in holography. The scalar diffraction formulae are most used ones in Digital holography. The research work reported in this thesis is also an attempt to derive out a new scalar diffraction formula for digital cylindrical holography. Hence it is worth discussing the various scalar diffraction theories, the approximation conditions and their significances.

Maxwell’s equations in terms of $E(r)$ and $H(r)$ can be written as

$$\nabla \cdot \epsilon E = 0 \quad (2.22)$$
$$\nabla \cdot \mu H = 0 \quad (2.23)$$
$$\nabla \times H = \epsilon \frac{\partial E}{\partial t} \quad (2.24)$$
$$\nabla \times E = -\mu \frac{\partial H}{\partial t} \quad (2.25)$$

where,

$\vec{E} \rightarrow$ electric field $(V/m)$
$\vec{H} \rightarrow$ magnetic field $(A/m)$
$\epsilon \rightarrow$ permittivity $(F/m)$
$\mu \rightarrow$ permeability $(H/m)$

The field vectors $E$ and $H$ both are functions of position $(x, y, z)$ and time $t$. As seen from Equations (2.22 and 2.25), Maxwell’s equations relate the field vectors by means of simultaneous differential equations. On elimination we obtain differential equations which each of the vectors must satisfy separately. For this, we apply the $\nabla \times$ operation to the left and right sides of Equation (2.25).

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \quad (2.26)$$

Let the propagation medium be linear, isotropic, homogeneous and nondispersive. Substituting the two Maxwell’s equations for $E$, Equations (2.22 and 2.25) into Equation (2.26) yields

$$\nabla^2 E - \epsilon \mu \frac{\partial^2 E}{\partial t^2} = 0 \quad (2.27)$$
The permeability and permittivity $\mu$ and $\epsilon$ are related with the wave velocity $v$ and refractive index $n$ as follows

$$v = \frac{c}{\sqrt{\mu \epsilon}} \quad (2.28)$$

where, $c$ is the velocity of light in vacuum.

$$n = \frac{c}{v} \quad (2.29)$$

Again, from Equation (2.28) and Equation (2.29) it could be derived that

$$n = \frac{1}{\sqrt{\mu \epsilon}} \quad (2.30)$$

Substituting Equations (2.28, 2.29 and 2.30) in Equation (2.27) yields,

$$\nabla^2 E - \frac{n^2}{c^2} \frac{\partial^2 E}{\partial t^2} = 0 \quad (2.31)$$

In the similar way we can obtain an equation for $H$ alone, which can be written as follows

$$\nabla^2 H - \frac{n^2}{c^2} \frac{\partial^2 H}{\partial t^2} = 0 \quad (2.32)$$

Equation (2.31) and Equation (4.3) are the standard equations of electromagnetic wave motion propagating with a velocity $c$.

Since the same vector wave equation is obeyed by both $\vec{E}$ and $\vec{H}$, it is possible to summarize the behavior of all components of $\vec{E}$ and $\vec{H}$ ($E_x, E_y, E_z, H_x, H_y, H_z$) through a single scalar wave equation.

$$\nabla^2 V(x, y, z, t) - \frac{n^2}{c^2} \frac{\partial^2 V(x, y, z, t)}{\partial t^2} = 0 \quad (2.33)$$

Hence, if the medium of propagation is linear, isotropic, homogeneous and nondispersive, all components of electric and magnetic field behave identically and their behavior is fully described by a single scalar wave equation as shown in Equation (2.33). However there is coupling between the components of electric and magnetic field at the boundaries. Hence even if the medium is homogeneous, the use of scalar theory entails some degree of error. But the error will be small and satisfactory results could be obtained, if the boundary conditions have effect over an area that is a small part of the area through which a wave may be passing. The wave propagation very well satisfies this condition in this research work and hence we turn our interest towards the scalar wave equation. The
scalar wave equation can still be simplified on inducing certain approximating conditions. These approximated equations are integral equations and are much easy for numerical evaluation. These approximated scalar wave equations are generally known as the scalar diffraction theories. The following explains the various diffraction theories and their approximating conditions.

For a monochromatic wave, the scalar field may be written explicitly as,

$$V(x, y, z, t) = U(x, y, z)e^{-i2\pi\nu t}$$  \hspace{1cm} (2.34)$$

where,

$$U(x, y, z) = U(P) = A(x, y, z)e^{i\phi(x,y,z)}$$  \hspace{1cm} (2.35)$$

where $A(x, y, z)$ and $\phi(x, y, z)$ are the amplitude and phase, respectively, of the wave at position $(x, y, z)$. $\nu$ is the frequency of the propagating wave. If this scalar field represents a propagating optical field, then it must satisfy the scalar wave equation represented in Equation (2.33) at each source free point. The complex function $U(x, y, z)$ serves as an adequate description of the wave, since the time dependence is known a priori. Accordingly, when Equation (2.34) is substituted in Equation (2.33) it follows that $U(x, y, z)$ shown in Equation (2.35) must obey the time-independent equation.

$$(\nabla^2 + k^2)U = 0$$ \hspace{1cm} (2.36)$$

where

$$k = 2\pi n \frac{\nu}{c} = \frac{2\pi}{\lambda}$$ \hspace{1cm} (2.37)$$

This relation shown in Equation (2.36) is known as the Helmholtz Equation. It can be very well stated that the complex amplitude of any monochromatic optical disturbance propagating in vacuum ($n = 1$) or in a homogeneous dielectric medium ($n > 1$) must obey Equation (2.36). The Helmholtz equation is the starting point for the derivation of the fast calculation formulas reported in this research work.

Before exploring the different diffraction theories, it is worth introducing the concept of diffraction. Diffraction is a phenomenon of considerable importance in the fields of physics and engineering whenever wave propagation is involved. Sommerfeld defined diffraction as “any deviation of light rays from rectilinear paths which cannot be interpreted as reflection or refraction” [40]. In 1665, the first account of diffractive phenomena was published by Grimaldi when he observed the shadow resulting from an aperture in an opaque screen illuminated by a light source. He observed that the transition from light to shadow was gradual rather than sharp. Sommerfeld’s definition implies that diffraction only applies to light rays. In reality, diffraction occurs with all types of waves.
including electromagnetic, acoustic, and water waves, and is present at all frequencies. The content of this thesis deals exclusively with electromagnetic radiation at optical frequencies.

Diffraction was initially considered to be a nuisance when designing optical systems because diffraction at the apertures of an optical imaging system is often the limiting factor in the system's resolution. However, by the mid 1900's, methods and devices utilizing the effects of diffraction began to emerge. Examples include analog holography, synthetic aperture radar, computer-generated holograms, digital holography and kinoforms, (also known as diffractive optical elements). As mentioned earlier, among these Digital holography is the main topic of this thesis.

The propagation of waves can often be described by rays which travel in straight lines (geometric optics). However, the behavior of wave fields encountering obstacles cannot be described by rays. Some of the wave encountering an obstacle will deviate from its original direction of propagation causing the resulting wave field to differ from the initial field at the obstacle. This is called diffraction. In other words “diffraction is a general characteristic of wave phenomena occurring whenever a portion of a wavefront be it sound, matter wave or light obstructed in some way”. Classic examples include diffraction of light from a knife’s edge and a wave field passing though an aperture in an opaque screen.

2.3.1.1 Huygens Fresnel Principle

The initial step in the evolution of a theory that would explain diffraction was made by Christian Huygens in the year 1678. Huygens expressed an intuitive conviction that if each point on the wavefront of a light disturbance was considered to be a new source of “secondary” spherical disturbance, then the wavefront at a later instant could be found by constructing the “envelope” of the secondary wavelets. But the technique ignores most of the secondary wavelets, retaining only that portion common to the envelope. As a result of this inadequacy, Huygens principle by itself was unable to account for the details of diffraction process.

The difficulty was resolved by Fresnel by the addition of the concept of interference. The corresponding Huygens-Fresnel Principle states that “every unobstructed point of wavefront, at a given instant, serves as a source of spherical secondary wavelets. The amplitude of the optical field at any point beyond is the superposition of all these wavelets (considering amplitudes and relative phases)”. Fresnel was able to calculate the distribution of light in diffraction patterns with excellent accuracy. The calculations are worked out by Hect [41]. Huygens-Fresnel Principle had a few short comings. First
of all the whole thing is rather hypothetical. Again, according to the principle at any instant every point on the primary wavefront is envisioned as a continuous emitter of spherical secondary wavelets. But if each wavelet radiated uniformly in all directions, in addition to the generating and ongoing wave, there would also be a reverse wave traveling back toward the source. No such wave is found experimentally.

The ideas of Huygens and Fresnel were put on a firm mathematical foundation by Gustav Kirchhoff. He showed that the amplitudes and phases ascribed to the secondary sources by Fresnel were indeed logical consequences of wave nature of light. He developed his rigorous theory based directly on the solution of Helmholtz wave equation Equation (2.36) using Green’s theorem. The complete derivation is given by Born and Wolf [6]. Accordingly the complex amplitude \( U(P) \) defined in Equation (2.35) is given by

\[
U(P) = \frac{1}{4\pi} \int \int_S \left[ U \frac{\partial}{\partial n} \left( \frac{e^{ikS}}{S} \right) - \frac{e^{ikS}}{S} \frac{\partial U}{\partial n} \right] \, dS
\]  

(2.38)

where \( S \) is the field boundary.

Thus Kirchhoff showed that, Huygens-Fresnel principle is an approximate form of a certain integral theorem which expresses the solution of the homogeneous wave equation at an arbitrary point in the field, in terms of the values of the solution and its first derivatives at all points on the arbitrary closed surface surrounding \( P \). Equation (2.38) is one form of the integral theorem of Helmholtz and Kirchoff. This integral theorem embodies the basic idea of Huygens-Fresnel principle but the laws governing the contributions from different elements of the surface are more complicated than Fresnel assumed. Kirchoff showed that, in many cases the theorem can be reduced to an approximate more simpler form. This resulted in the Kirchoff diffraction theory.

### 2.3.1.2 Kirchoff Diffraction Theory

Kirchhoff accordingly adopted the following assumptions to the problems.

1. Across the surface \( A \), the field distribution \( U \) and its derivative \( \frac{\partial U}{\partial n} \) are exactly the same as they would be in the absence of the screen.

2. Over the portion of \( B \), that lie in the geometrical shadow of the screen, the field distribution \( U \) and its derivative \( \frac{\partial U}{\partial n} \) are identically zero.

These conditions are commonly known as Kirchhoff boundary conditions. The first allows us to specify the disturbance incident on the aperture by neglecting the presence of the screen. The second allows us to neglect all of the surface integration except that
portion lying directly within the aperture itself. Accordingly the final expression for Kirchoff’s diffraction theory turned to be as shown below

\[ U(P) = -\frac{iA}{2\lambda} \int \int_A \frac{e^{ik(r+s)}}{rs} \left[ \cos(n,r) - \cos(n,s) \right] dS \]  \hspace{1cm} (2.39)

The detailed derivation of the result shown in Equation (2.39) is given by Goodman [35]. This result, which applies only for an illumination consisting of a single point source, is commonly known as the Fresnel-Kirchhoff diffraction formula. It allows one to calculate the optical disturbance at a point in space due to a diffracting object.
Kirchhoff mathematical development demonstrated that Huygen-Fresnel assumptions were in fact natural consequence of wave nature of light.

There are certain internal inconsistencies in this theory also. It is a well known theorem of Potential theory that if a two dimensional potential function and its derivative vanish together along any finite curve segment, then that potential function must vanish over the entire plane. Similarly, if a solution of the three dimensional wave equation vanishes on any finite surface element, it must vanish in all space. Thus the Kirchhoff’s two boundary conditions imply that the field is identically zero everywhere behind the aperture, a result that contradicts the physical situation. A further indication of these inconsistencies is the fact that the Fresnel-Kirchhoff diffraction formula can be shown to fail to reproduce the boundary condition as the observation point approaches the screen or aperture.

2.3.1.3 Rayleigh-Sommerfeld Diffraction Formula

The inconsistencies of the Kirchhoff theory were removed by Sommerfeld, who eliminated the necessity of imposing boundary values on both the disturbance and its normal derivative simultaneously.

Suppose the Kirchhoff theory was modified in such a way that, either $U$ or $\frac{\partial U}{\partial n}$ vanishes over the entire surface $B$, and not both. Then the necessity of imposing simultaneous boundary conditions on $U$ and $\frac{\partial U}{\partial n}$ would be removed, and hence the inconsistencies eliminated. Sommerfeld pointed out that Greens function with the required property do indeed exist.

Accordingly the Kirchhoff boundary condition may now be applied to $U$ alone (not $\frac{\partial U}{\partial n}$), which yields the following result

$$U(P) = -\frac{iA}{2\lambda} \int_A \frac{e^{ik(r+s)}}{rs} [\cos(n, r)]dS$$ (2.40)

This expression is known as Rayleigh-Sommerfeld diffraction formula. It yields wonderful results and has also removed the inconsistencies suffered by Fresnel-Kirchhoff. This formula is also not usually used in digital holography due to the complexity in numerical evaluation. It should be noted that in Kirchhoff and Sommerfeld theories, light is treated as a scalar phenomenon; i.e. only the scalar amplitude of one transverse component of either the electric or the magnetic field is considered. Any other component of interest can be treated independently in the similar manner. Such an approach entirely neglects the fact that the various components of electric and magnetic field vectors are coupled through Maxwell’s equations and cannot be treated independently. But experiments in
the microwave regions [42] have shown that scalar theory yields very accurate results if two conditions are met

1. The diffracting aperture must be large compared with the wavelength.
2. The diffracted fields must not be observed too close to the aperture.

Born and Wolf [6] have presented a complete discussion on the applicability of scalar diffraction.

The vectorial nature of the fields must be taken into account if reasonably accurate results are to be obtained. Vectorial generalizations of diffraction theory do exist. The first satisfactory one was proposed by Kottler [43]. The first truly rigorous solution of a diffraction problem was given in 1896 by Sommerfeld [44]. These theories are of no interest with regard to the work in this project.

2.3.1.4 Convolution Integral

It is also possible to formulate scalar diffraction theory in a framework that closely resembles the theory of linear, invariant systems. Accordingly the Rayleigh-Sommerfeld formula given by Equation (2.40) can also be expressed as

\[
U(P) = \frac{iA}{2\lambda} \iint_A U(P_0) \frac{e^{ikr}}{r} \cos(\theta) \mathrm{d}s
\]  

where \( \theta \) is the angle between the vectors \( n \) and \( r \). Now, Equation (2.41) is no more than a superposition integral. To make the point clear, Equation (2.40) can be re-written as

\[
U(P) = \iint_A h(P, P_0)U(P_0) \mathrm{d}s
\]  

where \( h(P, P_0) \) is given explicitly by

\[
h(P, P_0) = \frac{iA}{2\lambda} \iiint_A \frac{e^{ikr}}{r} [\cos(\theta)] \mathrm{d}\theta
\]

Equation (2.42) can be calculated using a convolution operation. The primary ingredient required for such a result is the property of linearity. The function \( h(P, P_0) \) is called as the point response function or the impulse response function. If the system is space-invariant, then Fast Fourier Transform can be used to evaluate this equation. Hence the numerical computation becomes more fast and easy. Hence this formula holds an
important place in Digital holography especially when the propagation distances are very small. Thus Huygens-Fresnel principle is nothing but a convolution integral.

### 2.3.1.5 Angular Spectrum of plane waves

Another formulation of the scalar diffraction theory in the framework of linear invariant systems theory is the *angular spectrum of plane waves*. It is very important to discuss this method because the research work in this thesis is based on this method. Hence the following discusses this in detail.

Let us consider the same situation where the wave field $U(x, y, z)$ travels in the $z$ direction. The wavefield is assumed to have a wavelength $\lambda$ such that $k = 2\pi/\lambda$. Let $z = 0$ initially. The 2-D Fourier representation of $U(x, y, 0)$ is given in terms of its Fourier transform $A(f_x, f_y, 0)$ by

$$U(x, y, 0) = \int_{-\infty}^{\infty} A(f_x, f_y, 0) e^{i2\pi(f_xx + f_yy)} df_x df_y$$  \hspace{1cm} (2.44)

where

$$A(f_x, f_y, 0) = \int_{-\infty}^{\infty} U(x, y, 0) e^{-i2\pi(f_xx + f_yy)} dx dy$$  \hspace{1cm} (2.45)

Including time variable to the integrand of Equation (2.45) gives $A(f_x, f_y, 0) e^{i2\pi(f_xx + f_yy + ft)}$. This represents a plane wave at $z = 0$ propagating with direction cosines $(\alpha, \beta, \gamma)$. Such a plane wave has a complex representation of the form

$$p(x, y, z, t) = e^{i(kr - 2\pi\nu t)}$$  \hspace{1cm} (2.46)

where $r = x\hat{x} + y\hat{y} + z\hat{z}$ is the position vector and $\vec{k} = 2\pi \lambda (\alpha\hat{x} + \beta\hat{y} + \gamma\hat{z})$. The direction cosines are interrelated through

$$\gamma = \sqrt{1 - \alpha^2 - \beta^2}$$  \hspace{1cm} (2.47)

Thus across the plane $z = 0$ the complex exponential function $e^{i2\pi(f_xx + f_yy + ft)}$ can be regarded as representing a plane wave propagating with direction cosines $\alpha = \lambda f_x$, $\beta = \lambda f_y$, $\gamma = \sqrt{1 - (\lambda f_x)^2 - (\lambda f_y)^2}$. In the Fourier decomposition of $U$, the complex amplitude of the plane-wave component with spatial frequencies $(f_x, f_y)$ is simply $A(f_x, f_y; 0)$, (with the time components discarded) evaluated at $(f_x = \alpha/\lambda, f_y = \beta/\lambda)$. 

For this reason the function
\[
A(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0) = \int_\infty^{-\infty} U(x, y, 0) e^{-i2\pi(\frac{\alpha x}{\lambda} + \frac{\beta y}{\lambda})} dx dy
\]  
(2.48)

is called as the **angular spectrum** of the disturbance \( U(x, y, z) \).

Consider now the wavefield \( U(x, y, z) \) parallel to \((x, y)\) but at a distance \(z\) from it given by
\[
U(x, y, z) = \int_\infty^{-\infty} A(f_x, f_y; z) e^{i2\pi(f_x x + f_y y)} df_x df_y
\]  
(2.49)

Let the function \( A(f_x, f_y, z) \) represents the angular spectrum of \( U(x, y, z) \). That is
\[
A(f_x, f_y; z) = \int_\infty^{-\infty} U(x, y, z) e^{-i2\pi(f_x x + f_y y)} dx dy
\]  
(2.50)

Now if the relation between \( A(f_x, f_y, 0) \) and \( A(f_x, f_y, z) \) can be found, then the effects of the wave propagation on the angular spectrum can be determined. To find the relation, let us consider the fact that, \( U(x, y, z) \) satisfies the Helmholtz equation at all source-free points namely,
\[
\nabla^2 U(x, y, z) + k^2 U(x, y, z) = 0
\]  
(2.51)

Substitution of \( U(x, y, z) \) from Equation (2.50) into Equation (2.51) yields
\[
\int_\infty^{-\infty} \left[ \frac{d^2}{dz^2} A(f_x, f_y; z) + (k^2 - 4\pi^2(f_x^2 + f_y^2))A(f_x, f_y; z) \right] e^{i2\pi(f_x x + f_y y)} df_x df_y = 0
\]  
(2.52)

This is true for all the waves only if the integrand is zero.
\[
\frac{d^2}{dz^2} A(f_x, f_y; z) + (k^2 - 4\pi^2(f_x^2 + f_y^2))A(f_x, f_y; z) = 0
\]  
(2.53)

An elementary solution to this differential equation can be written of the format
\[
A(f_x, f_y, z) = A(f_x, f_y, 0) e^{i\mu z}
\]  
(2.54)

where
\[
\mu = \sqrt{k^2 - 4\pi^2(f_x^2 + f_y^2)}
\]  
(2.55)
or in terms of direction cosines the solution can be written as

$$A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z\right) = A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right) e^{i\mu z} \quad (2.56)$$

where

$$\mu = \frac{2\pi}{\lambda} \sqrt{1 - \alpha^2 - \beta^2 z} \quad (2.57)$$

These results demonstrate that when the direction cosines \((\alpha, \beta)\) satisfy

$$\alpha^2 + \beta^2 < 1 \quad (2.58)$$

i.e., when \(\mu\) is real, the effect of propagation over distance \(z\) is simply a change of the relative phases of the various components of the angular spectrum by a phase factor \(e^{i\mu z}\). Since each plane wave component propagates at a different angle, each travels a different distance between two parallel planes, and relative phase delays are thus introduced. Plane wave components satisfying this condition are known as \textit{homogeneous waves}.

However when \((\alpha, \beta)\) satisfy

$$\alpha^2 + \beta^2 > 1 \quad (2.59)$$

then \(\alpha\) and \(\beta\) are no longer interpretable as direction cosines. The square root in Equation (2.57) and Equation (2.55) becomes imaginary and hence the Equation (2.54) becomes

$$A(f_x, f_y; z) = A(f_x, f_y; 0) e^{-\mu z} \quad (2.60)$$

Since \(\mu\) is a positive real number, the wave components are strongly attenuated by the propagation in the \(z\)-direction. They are called as \textit{evanescent waves}. These evanescent waves carry no energy from the aperture.

Hence knowing \(A(f_x, f_y, z)\) in terms of \(A(f_x, f_y, 0)\) allows us to find the wavefield at \((x, y, z)\) by using Equation (2.60) in Equation (2.49);

$$U(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(f_x, f_y; 0) e^{iz \sqrt{k^2 - 4\pi^2 (f_x^2 + f_y^2)}} e^{i2\pi(f_x x + f_y y)} df_x df_y \quad (2.61)$$

Thus, if \(U(x, y, 0)\) is known, \(A(f_x, f_y, 0)\) can be computed, followed by the computation of \(U(x, y, z)\). The limits of integration in Equation (2.61) can be limited to a circular region given by

$$4\pi^2 (f_x^2 + f_y^2) \leq k^2 \quad (2.62)$$
Background

provided the distance $z$ is at least several wavelengths long so that the evanescent waves may be neglected. Under these conditions, Equation (2.54) shows that wave propagation in a homogeneous medium is equivalent to a linear 2-D spatial filter with the transfer function given by

$$H(f_x, f_y) = e^{iz\sqrt{k^2-4\pi^2(f_x^2+f_y^2)}}$$  \hspace{1cm} (2.63)

or, in terms of direction cosines as

$$H(\frac{\alpha \lambda}{x}, \frac{\beta \lambda}{x}) = e^{iz\frac{2\pi}{\lambda} \sqrt{1-\alpha^2-\beta^2}}$$  \hspace{1cm} (2.64)

Equation (2.61) can also be represented in another form using the wavenumber, which provides more insight and easy understanding. For this, the relations $f_x = 2\pi k_x$, $f_y = 2\pi k_y$ and $k_z = \sqrt{k^2-k_x^2-k_y^2}$ are used in Equation (2.61) which reduces to

$$U(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(k_x, k_y, 0)e^{ik_z}e^{(ik_x+ik_y)d}dk_xdk_y$$  \hspace{1cm} (2.65)

Propagation of a wavefield in the $z$-direction in a source-free space is correctly described by the propagation of the angular spectrum in the near field as well as the far field. This is also often described as wave propagation in spectral domain. Two other ways to characterize such propagation are in terms of the Fresnel and Fraunhofer approximations which will be discussed in the next section.

Let $F$ and $F^{-1}$ denote the forward and inverse Fourier transform operators respectively. In terms of these operators, Equation (2.61) can be represented as

$$U(x, y, z) = F^{-1}[F[U(x, y, 0)]e^{ik_z\sqrt{1-\alpha^2-\beta^2}]}$$  \hspace{1cm} (2.66)

The above equations show that the numerical computation of the formula is also very easy and can be done using Fast Fourier transform if the shift invariant property exists. Hence this formula is more frequently used in digital holography especially when working in the near field. The research work reported in this thesis is to develop the same formula for wave propagation in cylindrical and spherical co-ordinates and to apply it to computer generated holography.
2.3.1.6 Summary

The various theories that explain wave propagation and the corresponding formulas were discussed starting from the very basic electromagnetic theory. Accordingly electromagnetic field at any point in space is defined by the Maxwell’s equations as seen from Equations (2.22 - 2.25). Light is also electromagnetic in nature and hence the disturbance (complex amplitude) of a propagating light field can be determined anywhere in space and at any time by the Maxwell’s equations. Numerically evaluating the complex amplitude using Maxwell’s equation is very difficult due its vectorial nature and also the discretization errors that creep in. But when the medium of propagation becomes, linear, isotropic, homogeneous and nondispersive the Maxwell’s equations can be greatly simplified to get rid of the vectorial nature. This gives rise to the scalar wave equation as seen from Equation (2.31). Experiments in digital holography satisfy these requirements and hence scalar diffraction theories are the mostly used ones to simulate wave propagation in digital holography. The scalar wave equation which is a differential equation can also be expressed as an integral equation based on a particular Green’s function. It also turns out that the Huygens-Fresnel postulate on diffraction can be mathematically expressed using this integral equation as seen from Equation (2.40). These non-vectorial integral equations constitute the Scalar diffraction theory. These equations can be further approximated based on the propagation distance which results in the Fresnel and Fraunhofer diffraction theories. These approximated theories reduce the computation complexity greatly and also fit well into most practical situations. Hence the Fresnel and Fraunhofer diffraction theories are the mostly used ones in digital holography. But they can be used with FFT only when the object surface and hologram surface are parallel to each other and perpendicular to the optical axis. There can be situations where the hologram surface is tilted or curved to the object surface. In these situations, only direct integration is possible which is very time consuming. This work is an attempt to device a wave propagation formula which can use FFT even when the hologram surface is a curved (cylindrical) one. All the scalar diffraction theories explained in this chapter define wave propagation in real space except the angular spectrum formula Equation (2.61) which defines propagation in spectral domain. This formula can also be evaluated by FFT, only if the object and hologram surface are plane and parallel to each other i.e., are shift-invariant. The work reported in this thesis is also to device a spectral propagation formula where the hologram surface is cylindrical and spherical but still could use FFT.

2.3.2 Methods in Holography

Instead of optical recording, the hologram associated with the wavefront representing the object is generated by employing different computational techniques and numerical
approaches by mathematically simulating the optical wave propagation. An ideal CGH should achieve complex light modulation at a high diffraction efficiency and precise reconstruction of the target image. The CGH’s outperform conventional refractive and diffractive components as a consequence of their ability to create any desired wavefront and thus to modify the input wavefront with much better flexibility \[45\]. For this reason CGHs find a wide range of application as display elements, optical interconnects, aberration compensators in optical testing, spatial filters for optical signal processing and computing, beam manipulators and array generators etc. CGH’s can be considered as thin optical elements with a complex amplitude transmittance. However, in many cases, they are phase only elements \[46\]. There are different classification of CGHs depending on the complex amplitude representation on the recording media (binary, phase, amplitude and combined phase-amplitude media), and the encoding method \[38\]. The algorithm to form a CGH is chosen according to the desired image characteristics and the associated computational complexity. Analytical approaches such as phase-detour method, kinoform method, double or multiple phase methods, explicit spatial carrier methods, 2-D simplex representation, representation by orthogonal and bi-orthogonal components, coding by symmetrization, etc., can be used for computing digital holograms \[38\]. There are cell-oriented and point-oriented methods. In cell-oriented CGH’s the hologram plane is divided into small resolution elements. The number of resolution cells needed depends on the complexity of the wavefront that is to be produced. Iterative approaches such as iterative Fourier transform algorithm \[47\], direct binary search \[48\], simulated annealing \[49\] have been proposed and used. These methods are computationally demanding.

However, CGHs which are intended for dynamic displays need faster algorithms. It is difficult to realize SLM’s which can provide the desired complex phase \[50\]. SLM’s with only binary modulation are particularly desirable for display of CGHs. Computer generated binary reflection holograms may be displayed using micro mirror devices (DMD) \[51\]. The SLM properties are crucial for the quality of the optical reconstruction of digital holograms. A comparison of the optical reconstruction of phase and amplitude holograms by different modulators in terms of diffraction efficiency and recovery quality is presented by Kohler et al. \[52\]. A Fourier transform based algorithm for fast calculation of diffractive structures, which permits image reconstruction on cylindrically and spherically curved surfaces, is developed by Sando et al. \[11\]. Another popular approach is to calculate the CGH as a superposition of analytic distributions by decomposing the object surface into a certain number of discrete independent point sources, line segments or higher-order image elements. The modeled underlying physical phenomenon is the interference between the light waves coming from the analytically defined holoprimitives.
constructing the object and the reference wave to form the resulting complex amplitude distribution on the hologram plane [53]. Hardware [54] and look-up table based computations were also proposed [55]. Representation of image elements at different locations by scaling and translation of similar elemental diffractive structures permits fast updating of the CGH by the so called incremental computing method [56]. Real color fractional Fourier transform holography is proposed by Jin et al. [57].

2.3.3 Sampling and Quantization

Sampling and quantization are inevitable when discretising any data. Sampling and quantization should be done properly to avoid any loss of interesting information and hence should be handled with a lot of care. Digital holography is no exception to this and discretization issues hold an important role in the process. Discretization issues like sampling and quantization are old topics, explored in detail with electrical signals and has well developed general theorems for loss free discretization. However diffraction related signals has additional special features and hence the general approaches are not efficient nor adequate enough. Hence discretization of diffraction related signals should be done by exploiting their special characteristics. This is an interesting and fruitful area of research which has improved the speed and efficiency of diffraction calculations. This section will review through the special characteristics of diffraction signals and their sampling and quantization requirements.

2.3.3.1 Sampling

A signal can be space- or band-limited but never both. For optical signals, the so called Fresnel limited functions turned to be more convenient and efficient than the band-limited functions in terms of sampling and recover ability. Fresnel limited functions are defined to have finite extent in their Fresnel transform domain associated with the parameter . Such functions are not band-limited, however, they can be reconstructed from their samples taken at a rate. The proof of this result is given by Gori [58]. Another theorem proven by Gori [58] indicates that the Fresnel transform of a space-limited function can be fully recovered from its Fresnel domain samples. The same result was also proven later independently by Onural [59] who also stated the prefect reconstruction conditions for both band- and space-limited cases. In particular, it is shown that for Fresnel transform, full recovery of space-limited signals is possible even when sampled below Nyquist rate. It is also shown that it is possible to reconstruct objects from hologram samples obtained below the Nyquist rate. Real-time applications by considering finite number of samples and finite (nonimpulsive) area of the capturing
charge coupled devices (CCD) array elements were also discussed. Furthermore, the
effect of sampling in noisy conditions is also analyzed. Thus the possibility of full
recovery from under sampled holographic signal is observed.

The effects of the shape of the sensing elements and the overall array size to the CCD
captured optical data and subsequent digital reconstruction of off-axis holography are
examined by Kreis [39]. A frequency domain analysis of the overall transfer function is
carried out for both the planar and the spherical reference beam cases.

It is shown that neither band-, nor space-limited functions can be fully recovered from
their samples if the replicas of their Wigner distributions due to sampling do not overlap.
Several nonuniform sampling schemes have been suggested based on the observation that
the bandwidth of the object remains unchanged as a consequence of the all-pass nature
of the linear system that represents the diffraction. Another approach observes that
the information of interest in a hologram is carried in the complex envelope of the
fringe pattern and not in the carrier [60]. Based on this, Khare and George [60] have
suggested sampling the recorded hologram about twice the Nyquist rate for the object
(or baseband) signal. This may be regarded as a generalization in the shift-invariant
space. In the work by Liebling et al. [61], where the modulation is replaced by the Fresnel
transform, can be noted as well. Wavelets have inspired several interesting approaches in
the area of optical signal sampling and reconstruction. The diffraction integral is viewed
as a continuous wavelet transform by Onural [62]. The light field at different distances is
regarded as the result of an inner product of the light distribution at some initial plane
with scaled and shifted chirp functions. In contrast to conventional wavelet analysis,
these scaling functions however, are not limited in neither the spatial nor the frequency
domain. The transform has been named scaling chirp transform and shown to be valid
and reversible by Onural and Kocatepe [63]. A number of inversion formulas are provided
with a discussion on their redundancy and ways to possibly exploit this redundancy. For
fixed scale, the scaled and shifted chirp functions form a complete orthogonal set, while
they form a redundant frame over different scales. This also suggests a way to sample
the light field throughout the space by using scaled chirp expansions.

Some related wavelet-like functions, called chirplets have been suggested by Mann and
Haykin [64], and used for instantaneous frequency measurements. A chirplet is a compact
support signal with increasing (decreasing) frequency [64]. It is band and time localized
version of the scaling chirp function mentioned above. Chirplets are rather attractive
for representation of holograms since they have minimal energy spread for the Fresnel
transform in a similar sense as Gabor functions [61]. Methods for finding a sparse
chirplet signal representation were suggested by Qian et al. [65]
An interesting strategy to construct bases suitable for processing digital holograms is presented by Liebling et al. [61]. Based on the observation that digital holography tends to spread out sharp details such as object edges over the entire imaging plane, standard wavelets have been ruled out as directly applicable to holograms. Instead, a Fresnel transform is applied to a wavelet basis to simulate the propagation in the hologram formation process. In contrast to classical wavelets, where multi resolution spaces are generated through dilation of one single function, in the fresnelets case there is one generating function for each scale. B-spline biorthogonal wavelets have been used to construct the fresnelet dictionaries due to their excellent approximation characteristics and analytical expression in spatial domain. Subsequently, their Fresnel transform associated wavelets are derived explicitly [61]. Thus, this new diffracted basis can be used to analyze the light field distribution at some distance and once a decomposition is obtained, the field can be calculated immediately in the original (initial) plane.

Digital reconstructions of diffraction patterns or holograms require algorithmic digital implementations of the underlying continuous mathematical models which represent diffraction. Common implementations of the Fresnel case are either based on convolution, or on a single Fourier transformation [66]. Inevitably, either the kernel which represent the wave-propagation (diffraction), or its analytically known Fourier transform (the transfer function) should be discretized when the convolution is implemented digitally. This problem is in the focus of the paper [67] where some well known properties of the continuous Fresnel kernel, together with rather overlooked ones are presented. Furthermore, efficient computation of the exact Fresnel transform of some periodic input (object) functions at some specific discrete distances is given, too. Another observation is the perfectly discrete and periodic nature of the continuous Fresnel transform of periodic and discrete input functions for certain distances.

2.3.3.2 Quantization

From a theoretical point of view, the diffraction is an operation which disperses the information content of simple object patterns over the entire space. Therefore, it is quite immune to noise or loss of information. Reconstructions from partial holograms could be pretty much satisfactory, with some bearable quality degradation. Therefore, it is expected that grossly digitized holograms would still yield reasonable reconstructions. Indeed, this fact was utilized for the computer-generation of holographic masks, going all the way to binary holograms. It might be interesting to look at oversampled, but coarse digitized cases.
Mills and Yamaguchi [68] discuss the quantization effects in phase-shifting holography. It provides both numerical simulations and experimental quality assessment and concludes that, for both uniform (specular) and random (diffuse) objects a 4-bit quantization is sufficient to recognize the reconstructed objects and the difference between 6 and 8 bits is not perceivable. Above 4 bits, the effect of quantization on the reconstructed image quality seems to be independent of the object phase distribution. Neal and Gallagher [69] have demonstrated that the quality of the reconstructed images from recorded holograms is more influenced by the phase information than the magnitude information. The paper assumes, with relevant arguments, that the magnitude has a Rayleigh distribution, whereas the phase is uniformly distributed over the interval. Then, a solution for minimum-mean-squared-error quantizer in polar form is formulated and numerically solved for some quantization levels. The allocation of bits between phase and magnitude is discussed.

It is observed that even though the phase and magnitude are statistically independent, the optimum magnitude quantization scheme depends on the number of phase quantization levels. The effects of phase quantization in Fourier holography is discussed by Dalla and Lohmann [70]. It is concluded that phase quantization results in ghost images located at different depths. It is further concluded that these ghost images are less disturbing particularly for high-contrast images, due to their different depths. Nonuniform quantization through computing of complex numbers by employing nonuniform grid patterns over the complex plane is shown to be efficient for digital holograms with a reconstruction quality comparable to that obtained by quantization by the \(k\)-means algorithm [71]. Quantization issues associated with holographic signals are discussed by Naughton et al. [72]. It is shown that degradation in reconstructed image quality is minimal for 10 bits or more, and the distortion becomes severe below 5 bits. Numerical error plots together with reconstructed images are presented.

### 2.3.4 3D Display Devices

#### 2.3.4.1 Holographic display

If a hologram is illuminated with the reconstructing light which is same as the recorded original, any observer interacting with the reconstructed light will see the same scene as original. Therefore in principle, holography creates true 3-D images, with all correct color, depth, shape information and parallax relation. This broad sense of definition involves all classical holographic techniques where coherent light is used to record the complex valued wavefront via interference and other true 3-D imaging techniques like
ideal integral imaging [73]. However in digital holography, holograms are stored as digital images in the computer which can be streamed into any display devices one after the other. Hence a dynamic display device is required to realize the video capabilities of digital holography. Candidate technologies for holographic display units include dynamically writable/erasable chemical films [74], or on electronically controllable arrays of pixels that can alter the phase and amplitude of light passing through (or reflected by) them, called spatial light modulators (SLMs) [75]. Specific forms, like deflectable mirror array devices (DMADs) are also among potential technologies that can be adapted for 3-D display, which can also be considered as special forms of SLMs [51]. Currently, dynamic chemical film technology is not mature enough for acceptable performance. Unfortunately, the size, quality and geometries of SLMs are currently not sufficient for acceptable quality 3-D displays, either. However, it is expected that both technologies will develop in time to yield the desired quality. There are also other techniques which are based on interaction of light with acoustic signals. Some experimental holographic 3DTV systems usually choose to sacrifice from the ultimate true 3-D display quality, for example by eliminating vertical parallax, and thus achieve higher resolution and fidelity in other features, or reduction in computational complexity. Ability to steer light from each point of a display device to arbitrary directions provides solutions to the dynamic display problem. Speckle noise in case of coherent illumination is another disadvantage, and there are proposed techniques to cope with this problem [76].

2.3.4.2 Stereoscopic Displays

Past and present implementations of most 3DTV systems rely on stereoscopy, or multiview video. In these approaches, no attempt is made to duplicate the original optical field. Instead, two or more 2-D images are captured at slightly different viewing angles. The human visual system interprets the received images. 3-D perception relies on the processing of several depth cues. Older type systems require special goggles to direct different images to each eye; however, newer systems utilize autostereoscopic systems to guide different 2-D views to different angles [77]. Systems based on stereoscopic principles usually create a feeling like motion sickness especially when some associated alignments are not perfect. Signal processing issues related with such display schemes are discussed in the review paper by Isgro et al. [78]. While the stereoscopy-based techniques are the most popular 3-D imaging techniques to date, holography-based techniques will most likely be the ultimate choice for digital holographic in the future.
2.4 Conclusion

It is very clear from the above discussions that, very little amount of work has been done with regard to wave propagation from cylindrical surfaces. But the basic theories governing wave propagation from non-planar surfaces resembles its planar counter part. Hence it is possible that non-planar holographic techniques should fit well into the architecture of the existing techniques discussed above. With this motivation, it was planned to do some research on the fast computational method for wave propagation from non-planar surface and apply it to computer generated holography. Accordingly a plane wave decomposition method for cylindrical and spherical surface was proposed and demonstrated with results. However, from the above discussion it can also be understood that, realizing cylindrical and spherical computer generated holography has its own difficulties too. The major one being the unavailability of a suitable dynamic display device in cylindrical or spherical shape. But fold-able polymer display devices are out in the market and hence we are not far from the days to have cylindrical display devices that could display fringe pattern in high resolution. The other major problem is the optical arrangement needed to illuminate the whole hologram which is very complex. This greatly reduces the portability of the hologram.
Chapter 3

Diffraction Theories for Wave Propagation from non planar surfaces

3.1 Introduction

Electromagnetic wave propagation in three-dimensional space is defined by the scalar diffraction theories, provided the source is non-polarized and medium has uniform refractive index. These theories completely define wave propagation from the radiating(source) surface to the detection surface in space. Accordingly the Rayleigh-Sommerfeld diffraction formula \[35\] is the most common formula for simulating wave propagation in computer generated holography as explained in section(2.3.1.3). The Rayleigh-Sommerfeld formula can also be expressed as a convolution operation as explained in section(2.3.1.4). If the source(object) and detection(hologram) surfaces are plane and parallel to each other, then the convolution operation can be executed using FFT. Hence the convolution form is mostly used for computation in computer generated holography. Wave propagation can also be defined in the spectral domain as an angular spectrum of plane waves which provided the Angular Spectrum formula for calculation as explained earlier in section(2.3.1.5). Again here, if the object and hologram surfaces are plane and parallel, then the formula can be evaluated using FFT. Compared to the Convolution formula, the Angular spectrum formula offers fast calculation and less sampling requirements. Hence this is the most used formula for computer generated holography. These are the three main formulas used in computer generated holography or digital holography for hologram generation and reconstruction. Based on the distance of propagation
between the object and hologram surfaces the convolution and angular spectrum methods can be approximated and expressed in an other form. These formulas are known as the Fresnel diffraction formula and the Fraunhofer diffraction formula which preserve enough accuracy in the calculated wavefield. When the object surface and hologram surface are non planar or non parallel to each other, then the shift invariance does not hold true for the system. Hence FFT cannot be used and the calculation takes a long time. The most popular and useful non-planar surface for holography is the cylindrical hologram and spherical hologram. Hence these are the only surfaces considered for this research work. Usually cylindrical and spherical holograms are computed using the Rayleigh-Sommerfeld integral, which is expressed in cylindrical or spherical co-ordinates. The expression in cylindrical and spherical co-ordinate is obtained by applying the coordinate transformation operation on the Rayleigh-Sommerfeld diffraction formula for cartesian co-ordinates. The Rayleigh-Sommerfeld integral is evaluated using direct integration by trapezoidal or any other method and is very slow. However under special cases, FFT and other fast computation methods can also be used for the simulation of wave propagation for cylindrical and spherical CGH. Such fast computation formulas are the main interest in this research work. The aim in this research work is to develop a new and more efficient fast calculation formula. Hence this chapter will explain the formulas already reported for calculating wave propagation for cylindrical and spherical computer generated holography. Then the disadvantages suffered by the reported problems and the need for more efficient formulas will be presented. Finally, the ways to find a better formula and the expected advantages are described in detail.

3.2 Cylindrical CGH

When considering cylindrical CGH the task is to simulate wave propagation from all points on an object to a cylindrical hologram surface as shown in Figure. 3.1. For this the most general way to do is, by using the Rayleigh-Sommerfeld diffraction integral. The Rayleigh-Sommerfeld diffraction integral in cylindrical co-ordinates is given by Sando et al. [11] and can be described as follows

$$H(r, \theta, y) = \int \int \frac{O(r_0, \theta_0, y_0) \exp(iKL)}{L} dr_0 d\theta_0 dy_0$$  (3.1)

where,

$$L = \sqrt{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0) + (y - y_0)^2}$$  (3.2)
and, \( O(r_0, \theta_0, y_0) \) and \( H(r, \theta, y) \) denotes the object and hologram surface in cylindrical co-ordinates respectively. The above formula can be used for any arbitrary object, however the calculation is very slow due to direct integration. In order to speedup computation and use FFT many methods have been proposed. Yamaguchi et al. [16] used the segmentation approach. Here the cylindrical hologram surface is segmented into elemental planar segments. Each segment is considered to be plane and parallel to the object elemental surface. Now the system consists of the object and hologram as pairs of parallel and plane surfaces and hence FFT could be used for calculation. They used the Fresnel diffraction formula for calculation and could achieve high speeds. However since the cylindrical surface is approximated to be elemental plane surfaces, the calculated hologram is not accurate. Another approach is to make use of the shift invariance in rotation between a plane surface and cylindrical surface. When a plane surfaced object is placed in the center of a cylindrical hologram, and the object is rotated, the pattern in the hologram plane also rotates about the same amount without any change. Hence such a system has the privilege to use FFT for computing the wave field in the horizontal direction. However there is no shift invariance in the vertical direction and hence FFT cannot be used for the vertical direction. This method was used by Sakamoto et al. [12] to calculate a cylindrical CGH and the setup is shown in Figure 3.2.

The usage of FFT for both the horizontal and vertical direction was made possible by Sando et al. [11] using a brilliant idea. Instead of assuming the object to have arbitrarily distributed points, they assumed the object to have data points on a cylindrical plane that is concentric to the cylindrical hologram plane. A 3D object can also be modelled by using a set of such concentric cylindrical surfaces. Now the task is to calculate wave propagation from each object surface to the hologram surface. Since both the surfaces
are concentric cylinders a shift in vertical direction or rotation in horizontal direction will not affect the hologram pattern. Hence there is shift invariance in such a system and FFT can be used to computation in both the directions. Such a system is shown in Figure 3.3.

\[ p(\theta, y) = \frac{e^{ik\sqrt{r_0^2 + r^2 - 2r_0r\cos\theta + y^2}}}{\sqrt{r_0^2 + r^2 - 2r_0r\cos\theta}} \] (3.3)
Then the wave field is calculated on the cylindrical hologram plane using the point spread function as shown in equation (3.4)

\[ H(\theta, y) = \int \int O(\theta_0, y_0) p(\theta - \theta_0, y - y_0) d\theta dy \] (3.4)

For the use of FFT in calculation Equation (3.4) can be expressed as a convolution operation as shown in equation (3.5)

\[ H(\theta, y) = O(\theta_0, y_0) * p(\theta, y) \] (3.5)

The convolution operation converts into a multiplication in the Fourier space and hence equation (3.5) can be evaluated using two FFT operation and a multiplication and then followed by an IFFT operation. The computation is very fast and gives good results. However as seen from equation (3.1), the obliquity factor that is an integral part of the Rayleigh-Sommerfeld formula is neglected. This is done to achieve the form of convolution operation in Rayleigh Sommerfeld integral. The obliquity factor can be neglected in case of plane holograms considering small angles of propagation. However this not the case in cylindrical holography and hence produces errors. Hence the above mentioned method is not very efficient to generate computer generated cylindrical holography.

The above mentioned issue can be solved if a spectral wave solution is sought which is analogous to the angular spectrum of plane waves. For this the solution to the Helmholtz equation is to be found out by using the boundary conditions. Such a solution would inherently take care of the obliquity factor and hence will be more efficient. In addition the spectral propagation formula will also be fast in computation, because only two FFT operations and a multiplication are required. Moreover the spectral formula decomposes the wavefield into its spectrum and then calculates propagation and hence sampling becomes less. With all these benefits the spectral wave propagation formula for cylindrical computer generated holography becomes very essential.

### 3.3 Spherical CGH

Spherical holograms has the ability to reconstruct the complete object wave front for 360\(^\circ\) in both azimuth (horizontal) and zenith (vertical) directions. The plane and cylindrical holograms discussed above lack this ability and hence spherical holography is an important candidate for research and development. But due to the difficulties in optical
Diffraction Theories for Wave Propagation from non planar surfaces

recording and reconstruction setups, and due to the un-availability of fast computation formulas, spherical holography and spherical computer generated holography is not so popular. However with the advance in optical technology the difficulties are slowly disappearing. A report on hemispherical computer generated holography was reported by Rosen [15]. The system considered is shown in figure 3.4. Here the task was to simulate wave propagation from the arbitrary object points to the hemispherical hologram surface. As clear from the figure 3.4 the system does not have any shift invariance neither in horizontal nor in vertical direction. Hence wave propagation was simulated by evaluating the Fresnel integral directly which is a slow process. For the evaluation the hemispherical hologram surface was defined in Cartesian co-ordinate system.

![Figure 3.4: Hemispherical CGH - Schematic](image)

Wave propagation to spherical holograms can generally be defined by representing the Rayleigh-Sommerfeld diffraction formula in spherical co-ordinates. This allows the object and hologram surfaces also to be defined in spherical co-ordinates which makes the computation very easy. Such a system is shown in figure 3.5. The Rayleigh-Sommerfeld formula can be expressed in spherical co-ordinates as shown below in equation (3.6).

\[
H(r, \theta, \phi) = \int \int \frac{O(r_0, \theta_0, \phi_0) e^{ikL}}{L} d\theta_0 d\phi_0
\]  

(3.6)

where,

\[
L = \sqrt{r^2 + r_0^2 - 2rr_0(cos \theta cos \theta_0 cos(\phi - \phi_0) + sin \theta sin \theta_0)}
\]  

(3.7)

Here, \(H(r, \theta, \phi)\) and \(O(r_0, \theta_0, \phi_0)\) represent the hologram and object surfaces in spherical co-ordinates. By directly integrating the equation (3.6) using trapezoidal rule or any
other method, the wave field due to arbitrary object points can be calculated in a spherical surface. However the method is very slow due to the usage of direct integration.

Fast computation of spherical computer generated holograms were first reported by Tachiki et al. [14]. Here again, the object surface is assumed to be a spherical surface or a set of concentric spherical surfaces to the hologram surface as shown in figure 3.6. This will establish a shift invariant relation between the object and hologram surface and there by enabling fast computation using FFT. To use FFT the Rayleigh-Sommerfeld diffraction integral mentioned in equation (3.6) should be expressed as a convolution integral. For this the point spread function is to be found out. The point spread function is derived out by approximating the distance formula shown in equation (3.7) as shown below in equation (3.8).

\[ L = \sqrt{r^2 + r_0^2 - 2rr_0 \cos(\phi - \phi_0) \cos(\theta - \theta_0)} \]  

(3.8)

Using the above mentioned distance formula, the approximated point spread function for spherical wave propagation can be represented as shown below in equation (3.9)

\[ p(\theta, \phi) = \frac{e^{\sqrt{ik[r^2 + r_0^2 - 2rr_0 \cos(\phi) \cos(\theta)]}}}{\sqrt{ik[r^2 + r_0^2 - 2rr_0 \cos(\phi) \cos(\theta)]}} \]  

(3.9)

Substituting the above mentioned PSF into equation (3.6) we get...
Figure 3.6: Spherical CGH - Schematic of shift invariant system

$$H(r, \theta, \phi) = \int \int O(r_0, \theta_0, \phi_0)p(\theta - \theta_0, \phi - \phi_0)d\theta_0d\phi_0 \quad (3.10)$$

The integral equation (3.10) can be expresses as a convolution operation as shown below

$$H(\theta, \phi) = O(\theta_0, \phi_0) \ast p(\theta, \phi) \quad (3.11)$$

The above mentioned equation (3.11) calculates the wavefield due to the spherical object with radius $r_0$ at the hologram surface with radius $r$. The convolution operation can be executed using two FFT operations and a multiplication. Hence this formula is very fast in computation. However, as discussed earlier the formula suffers from approximations and hence not so accurate. More over the obliquity factor is discarded in the Rayleigh-Sommerfeld integral which generates a lot of error especially in this case of spherical holography where the diffraction angles are very large.

The in-efficiencies of the convolution method can be overcome by finding solutions to the Helmholtz wave equation for the spherical system. This formula will define wave propagation in spectral domain and hence can be referred as the spherical wave spectrum method. This method is also analogous to the angular spectrum method. The solution could be obtained by considering it as a boundary value problem on a spherical surface.

As mentioned the expected advantages of this formula will be i) less calculation time, ii) more accuracy and iii) less sampling. Hence finding such a formula is very essential and useful. Thus this research work attempts to achieve the same.

All the diffraction theories discussed in this chapter are presented in an abstract form as a table below as figure 3.7. This will help for a more precise understanding of the
diffraction theories and their advantages, disadvantages and differences compared to others.

3.4 Conclusion

This chapter presented a review of the diffraction formulas already reported for non-planar holography. The drawbacks of the reported methods and ways to solve the issue is also discussed.
### Diffraction Theories for CGH - Comparison

<table>
<thead>
<tr>
<th>Surface Type</th>
<th>Direct Integration</th>
<th>Convolution</th>
<th>Angular Spectrum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plane CGH</td>
<td>$H(x, y) = \int \frac{O(x_0, y_0) \exp(ikL)}{L} , dx , dy$</td>
<td>$H(x, y) = O(x_0, y_0) \otimes \text{PSF}$</td>
<td>$H(x, y) = \text{FFT}^{-1} \left[ \text{FFT} \left[ O(x_0, y_0) \right] \times TF \right] - \exp\left[ikz\sqrt{1 - (\lambda f_x)^2 - (\lambda f_y)^2}\right]$</td>
</tr>
<tr>
<td>Cylindrical CGH</td>
<td>$H(\theta, y) = \int \frac{O(\theta_0, y_0) \exp(ikL)}{L} , dx , dy$</td>
<td>$H(\theta, y) = O(\theta_0, y_0) \otimes \text{PSF}$</td>
<td>$\text{PSF} = \frac{\exp[ik(\sqrt{R^2 + r^2 - 2R\cos(\theta) + y^2})]}{(\sqrt{R^2 + r^2 - 2R\cos(\theta) + y^2})}$</td>
</tr>
<tr>
<td>Spherical CGH</td>
<td>$H(\theta, \phi) = \int \frac{O(\theta_0, \phi_0) \exp(ikL)}{L} , dx , dy$</td>
<td>$H(\theta, \phi) = O(\theta_0, \phi_0) \otimes \text{PSF}$</td>
<td>$\text{PSF} = \frac{\exp[ik\sqrt{R^2 + r^2 - 2R\cos(\phi)\cos(\theta)}]}{\sqrt{R^2 + r^2 - 2R\cos(\phi)\cos(\theta)}}$</td>
</tr>
</tbody>
</table>

**Figure 3.7:** Diffraction theories for CGH - Comparison
Chapter 4

Cylindrical Wave Propagation

4.1 Introduction

The phase and amplitude of light propagating from cylindrical surface varies in space (with time) in an entirely different fashion compared to that from a plane surface. But still they fit well into the framework of diffraction theories for wave propagation from plane surface [6]. Hence wave propagation from cylindrical surfaces has very interesting properties. Generally, the complex amplitude of a wave propagating from a source surface to an observation surface can be calculated using the diffraction theories as explained in the previous chapter. If the source and observation surface are planes parallel to each other, then these diffraction theories can be simplified a lot. Moreover since such a system is shift-invariant the calculation speed can be dramatically improved by using fast Fourier transform. But if the source or observation surface is curved or tilted with respect to each other, then the complex amplitude can be calculated only using direct integration, which is time consuming. But if both the source and observation surface are curved such that the shift-invariance still holds, then fast Fourier transform can be used. Such a situation occurs when both the object and observation surface are concentric cylindrical surfaces. The diffraction theories then have to be expressed in cylindrical coordinates \((r, \theta, h)\), for fast Fourier transform to be used. So the diffraction formulae that were devised in Cartesian coordinates (in the previous chapter) has to be devised in cylindrical coordinates. The two near field and less approximated calculation methods, a) convolution method and b) angular spectrum of plane waves method are of interest in cylindrical geometry. The convolution formula has already been devised for cylindrical coordinates and digital cylindrical holography based on convolution method has been demonstrated by Sando et al. [11]. But digital holography has not yet been demonstrated using the plane wave decomposition method (also known
as angular spectrum method), which is much faster than the convolution method. This work is an attempt to do the same. In other words this work is an attempt to devise a formula for wave propagation from cylindrical surface in spectral domain and apply it to digital holography.

In order to appreciate the usefulness and challenges in this, a very short review of the evolution of computer generated cylindrical holography is presented below. In the beginning days, computer generated holograms were made only on planar surface due to the existence of the shift invariance relationship which allowed using FFT for calculations. Hence computer generated holograms were usually made for geometries where the object and hologram surfaces are planar and perpendicular to the optical axis. The other reason was the availability of planar hologram recording and display devices. Later on the method was improved and fast computation schemes using FFT were also developed for non-shift invariant systems such as, a tilted plane geometry [79, 80]. But for a long time the cylindrical geometry was not considered for making a computer generated hologram. The main reason was the extreme difficulty in devising a fast computation method for a curved hologram surface.

Fast computation method is possible if the wave propagation is devised in cylindrical co-ordinates. For this the solution to the wave equation (Equation 2.27) is to be derived in cylindrical co-ordinates. The normal mode solutions to Equation (2.27) in cylindrical and spherical geometries were given by Stratton [81]. Berry [82] discusses on the propagation of cylindrical waves and the occurrence of evanescent waves. The construction of Greens function for cylindrical and spherical geometries was given by Marathay [83]. Even though the theories on cylindrical wave propagation were available, they were less attractive due to the unavailability of cylindrical shaped recording or display devices. However the recent developments in technology and the possibility of producing advanced display and recording devices, has made people focus on cylindrical geometries for digital holography. As a result, papers describing cylindrical computer generated holography started to appear in the recent five years. Sakamoto and Tobise [10] used the angular spectrum of plane waves method to generate a cylindrical hologram of a plane object. They employed the shift invariance in rotation between a planar and cylindrical surface and hence could use FFT. Then they improved on their method to generate the hologram of a volume object by slicing it into planar segments [12]. This took 2.76 hrs to calculate the hologram of a 13×13×13 mm object. Yamaguchi et al. [13] used the Fresnel transform and segmentation approach to generate cylindrical holograms. They approximated the cylindrical holographic surface into smaller plane surface to generate the hologram. Since it was a multiplexed hologram they had to use a large number of samples which demanded a calculation time of 81 hrs on a parallel computing machine.
for an object of size $15 \times 15 \times 15$ mm. They also developed a computer generated cylindrical rainbow hologram using the same method \cite{16}. The calculation time for the rainbow hologram was 45 min on a single computer, but sacrifices vertical parallax. Then Sando et al. \cite{11} took a very different and smart approach by considering the object also to be a cylindrical surface. Both the object surface and the hologram surfaces are concentric cylinders. The most important significance of this approach is that, the shift-invariance relation is preserved in both horizontal and vertical directions and hence FFT can be used. However their calculation method was based on wave propagation in spatial domain using convolution. This method uses three FFT calculations and is faster than all the other existing methods for cylindrical holography. However if a method could be devised using wave propagation in spectral domain using decomposition of plane waves theory, then it would be still faster because it uses only two FFT operations. So far no one has computed a cylindrical hologram by considering wave propagation from a cylindrical surface in spectral domain. This research work attempts to achieve the same using a spectral method in cylindrical coordinates. In other words, this method is the counter part of angular spectrum of plane waves method in Cartesian coordinates. For any spectral propagation method the most important task is to find the proper transfer function and then characterize it for proper sampling conditions and if possible approximate it for easy computation. Hence this chapter will explain in detail the derivation of the transfer function and also discuss the proper sampling conditions required for loss free numerical computation. In more general terms, this chapter will devise the plane wave decomposition method in cylindrical coordinates. This can also be considered as an improvement or extension of the work by Sando et al.\cite{11}.

### 4.2 Helical Wave Spectrum

Since the diffraction theory for propagation from cylindrical surface fits into the frame work of that of plane surface, the same procedure followed in the previous chapter (explaining wave propagation in rectangular coordinates) will be followed in this section. Accordingly we start with the scalar wave equation expressed earlier as Equation (2.27), but now in cylindrical coordinates$(r, \theta, y, t)$.

\begin{equation}
\nabla^2 E - \epsilon \mu \frac{\partial^2 E}{\partial t^2} = 0
\end{equation}

The cylindrical coordinate system is shown in Figure 4.1. In cylindrical coordinates the Laplace operator $\nabla^2$ is defined as
\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial y^2} \] \tag{4.2}

As explained in Chapter 2, if the medium of propagation is linear, isotropic, homogeneous and nondispersive, Equation (4.3) can be represented as a scalar equation. Hence we drop the vectorial nature of the equation and look for a scalar function \( p(r, \phi, y, t) \) as the solution. Hence Equation (4.1) can be expressed as

\[ \nabla^2 p - \epsilon \mu \frac{\partial^2 p}{\partial t^2} = 0 \] \tag{4.3}

The solution to the wave equation (Equation 4.3) can be found using separation of variables method. For this, the solution has to be written as a product of solutions of function of each coordinate and of time. That is

\[ p(r, \phi, y, t) = R(r) \Phi(\phi) Y(y) T(t) \] \tag{4.4}

Substituting the solution in Equation (4.3) and dividing out by \( R\Phi YT \) (for separation of variables), leads to

\[ \left( \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{r R} \frac{dR}{dr} + \frac{1}{r^2 \Phi} \frac{d^2 \Phi}{d\phi^2} \right) + \left( \frac{1}{Y} \frac{d^2 Y}{dy^2} \right) = \frac{1}{c^2 T} \frac{dT}{dt} \] \tag{4.5}

The terms in the first set of brackets depend only on the variables \( r \) and \( \phi \) and the second set of brackets only on \( y \) and the right hand side only on \( t \). Thus since \( r, \phi, y \) and \( t \) are all independent of each other, each of these terms must be equal to a constant. We
Cylindrical Wave Propagation

choose the following arbitrary constants, $k$ and $k_y$, satisfying the following equations.

\[
\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = -k^2 \tag{4.6}
\]

\[
\frac{1}{Y} \frac{d^2 Y}{dy^2} = -k_y^2 \tag{4.7}
\]

\[
\frac{1}{R} \left( \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + \frac{1}{r^2 \Phi} \frac{d^2 \Phi}{d\phi^2} = -k^2 + k_y^2 = -k_r^2 \tag{4.8}
\]

where the constant

\[
k_r = \sqrt{k^2 - k_y^2} \tag{4.9}
\]

Equation (4.8) can be written as

\[
\frac{r^2}{R} \left( \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + k_r^2 r^2 = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} \tag{4.10}
\]

The left hand side of Equation (4.10) is a function of $r$ alone and the right hand side of $\phi$ alone. Hence the right and left hand sides must be equal to constants. Choosing $n^2$ as one of the constants leads to

\[
\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = -n^2 \tag{4.11}
\]

and the left hand side turns out to be the Bessel’s equation,

\[
\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left( k_r^2 - \frac{n^2}{r^2} \right) R = 0 \tag{4.12}
\]

The solutions of Equation (4.12) are well known and are given by the Bessel functions of the first and second kinds $J_n(k_r r)$ and $Y_n(k_r r)$. $Y_n$ is also called as the Neumann function. The solution to Equation (4.12) uses these two independent functions with arbitrary constants $R_1$ and $R_2$

\[
R(r) = R_1 J_n(k_r r) + R_2 Y_n(k_r r) \tag{4.13}
\]

$J_n$ and $Y_n$ are called standing wave solutions of Equation (4.12) because of their asymptotic behavior. A linear combination of these functions is necessary for traveling wave solutions, and is given by the Hankel functions of the first and second kind.
\[ H_n^{(1)}(kr) = J_n(kr) + iY_n(kr) \]  
\[ H_n^{(2)}(kr) = J_n(kr) - iY_n(kr) \]

With the time dependence \( e^{-i\omega t} \), \( H_n^{(1)}(kr) \) corresponds to a diverging outgoing wave and \( H_n^{(2)}(kr) \) to an incoming converging wave. The general traveling wave solution is then

\[ R(r) = R_1 H_n^{(1)}(kr) + R_2 H_n^{(2)}(kr) \]

Similarly, since Equation (4.6), Equation (4.7) and Equation (4.11) are second order differential equations, each has a general solution with two arbitrary constants

\[ \Phi(\phi) = \Phi_1 e^{i n \phi} + \Phi_2 e^{-i n \phi} \]  
\[ Y(y) = Y_1 e^{i k_y y} + Y_2 e^{-i k_y y} \]  
\[ T(t) = T_1 e^{-i \omega t} + T_2 e^{-i \omega t} \]

with arbitrary constants \( \Phi_1, \Phi_2, Y_1, Y_2, T_1 \) and \( T_2 \). Further, the quantity \( n \) must be an integer because \( \Phi(\phi + 2\pi) = \Phi(\phi) \), and \( k = \omega/c \). Also we assume \( T_2 = 0 \) for the convention of time.

Now, we combine the solutions given by Equation (4.16), Equation (4.17), Equation (4.18) and Equation (4.19). There are six possible combinations with the two independent solutions for each coordinate.

\[ p(r, \phi, y, t) \propto H_n^{(1)-(2)}(kr) e^{\pm i n \phi} e^{\pm i k_y y} e^{-i \omega t} \]

All these six combinations can be included in the general solution by summing over all possible positive and negative values of \( n \) and \( k_y \) with arbitrary coefficient functions (functions of \( n, k_y \) and \( \omega \)) replacing the pairs of constants, \( Y_1, Y_2, \Phi_1, \Phi_2, R_1 \) and \( R_2 \). Thus the most general solution to Eq. (4.3) in the spectral domain is given by

\[ p(r, \phi, y, \omega) = \sum_{n=-\infty}^{\infty} e^{i n \phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ A_n(k_y, \omega) e^{i k_y y} H_n^{(1)}(kr) + B_n(k_y, \omega) e^{i k_y y} H_n^{(2)}(kr) \right] dk_y \]
where $A_n(k_y, \omega)$ and $B_n(k_y, \omega)$ are the arbitrary constants replacing the constants $Y_1, Y_2, \Phi_1, \Phi_2, R_1$ and $R_2$.

The time domain solution of the wave equation (Equation 4.3) can be obtained from the inverse Fourier transform.

\[
p(r, \phi, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(r, \phi, y, \omega) e^{-i\omega t} d\omega
\]

(Equation 4.22)

![Figure 4.2: Boundary conditions in cylindrical coordinates](image)

Equation (4.21) represents the complete general solution to the wave equation in a source-free region. In order to determine the arbitrary coefficients, boundary conditions are to be specified on the coordinate surfaces, for example, $r = constant$. Boundary conditions with $y = constant$ leads to discrete solutions in $k_y$ instead of continuous ones formulated above. The boundary condition on $r$ alone leads to the solution that suits the problem discussed in this research work. Hence we proceed in finding the solution by imposing the boundary condition on $r$.

Consider the case in which the boundary condition is specified at $r = a$ and $r = b$, as shown in Figure 4.2. In this case the sources are located in the two regions labeled $\Sigma_1$ and $\Sigma_2$. The homogeneous wave equation is valid in the annular disk region shown in Figure 4.2. In this region Equation (4.21) can be used to solve for the wavefield. The boundary conditions on the surfaces at $r = a$ and $r = b$ yield unique solution (for all values of $y$ and $\phi$). Two boundary conditions are necessary because there are two unknown functions, $A_n$ and $B_n$ in the equations. No part of the source region is allowed to cross the infinite cylinder surfaces defining the annular disk region.

The two parts to the solution of Equation (4.21) can be explained with respect to the two Hankel functions. The first term represents an outgoing wave expressed in
Equation (4.14) due to sources which must be on the interior of the volume of validity \((\sum_1)\), causing the waves to diverge outward. \(A_n\) provides the strength of these sources. The second Hankel function (Equation 4.15) represents incoming waves and is needed to account for the sources external to the annular region \((\sum_2)\). Similarly, \(B_n\) provides the strength of these sources.

![Figure 4.3: All sources outside boundary](image1)

![Figure 4.4: All sources inside boundary](image2)

Now, two other boundary conditions also arise which are shown in Figures 4.3 and 4.4 respectively. The first one is called the interior problem in which the sources are located completely outside the boundary surface \(r = b\) (Figure 4.3). The second boundary problem is called the exterior problem because the boundary surface \(r = a\) completely encloses all the sources (Figure 4.4). The research work reported in this thesis is also a problem of this kind. Hence we proceed discussing with only the second boundary value problem shown in Figure 4.4. Now the solution to Equation (4.21) is to be found out based on this boundary condition. It turns out that the second term in Equation (4.21) represents an in-coming wave which can not exist when all the sources are within the boundary. Thus we set the second coefficient function to zero i.e, \(B_n = 0\). Now the general solution becomes
\[ p(r, \phi, z, \omega) = \sum_{n=-\infty}^{\infty} e^{in\phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} A_n(k_y, \omega) e^{ik_y y} H_n^{(1)}(k_y r) \, dk_y \quad (4.23) \]

Now, if the wavefield on the boundary at \( r = a \) is specified then \( A_n \) can be determined and Equation (4.23) can be used to solve for the wavefield in the region from the surface at \( r = a \) to \( r = \infty \) (Figure 4.4). In this reported research work, the boundary surface at \( r = a \) constitutes the object surface whose wavefield is already known and the hologram is another surface that is exterior to \( r = a \). Hence this research work also demands a solution of the same kind. Hence now we proceed to determine the quantity \( A_n \) using the known boundary values.

Since the time dependence of the propagation is known a priori (for a monochromatic wave), the time component (\( \omega \), in spectral domain) can be neglected in Equation (4.23). It is also worth noting here that the wave equation with the time component dropped is nothing but the Helmholtz equation defined in Chapter 2 as Equation (2.36). Hence, in other words solution to Helmholtz equation is being found out as it was done in Chapter 2, but now in cylindrical coordinates. Hence Equation (4.23) reduces to

\[ p(a, \phi, y) = \sum_{n=-\infty}^{\infty} e^{in\phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} A_n(k_y) e^{ik_y y} H_n^{(1)}(k_y a) \, dk_y \quad (4.24) \]

Now, let us consider \( P_n(r, k_y) \) to be the two-dimensional Fourier transform in \( \phi \) and \( y \) in cylindrical coordinates of the wavefield defined at \( r \).

\[ P_n(r, k_y) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} p(r, \phi, y) e^{-in\phi} e^{-ik_y y} \, dy \quad (4.25) \]

The inverse relation for Eq. (4.25) is given by

\[ p(r, \phi, y) = \sum_{n=-\infty}^{\infty} e^{in\phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} P_n(r, k_y) e^{ik_y y} \, dk_y \quad (4.26) \]

where \( n \) can take only integer values because the cylindrical surface is a closed one in the circumferential direction. Comparing Equation (4.26) at \( r = a \) with Eq. (4.24) we get

\[ P_n(a, k_y) = A_n(k_y) H_n^{(1)}(k_y a) \quad (4.27) \]
Cylindrical Wave Propagation

Using Eq. (4.27) to eliminate $A_n$ in Eq. (4.24) yields

$$p(r, \phi, y) = \sum_{n=-\infty}^{\infty} e^{in\phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} P_n(a, k_y) e^{ik_y y} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} dk_y$$

(4.28)

where,

$$P_n(a, k_y) = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{-\infty}^{\infty} p(a, \phi, y) e^{-in\phi} e^{-ik_y y} dy$$

(4.29)

Equation (4.28) calculates the complex amplitude at any position $p(r, \phi, y)$, given the complex amplitude in another cylindrical surface $p(a, \phi, y)$ such that $r > a$. The spectral solution in Equation (4.28) is similar in form to the plane wave expansion (angular spectrum of plane waves) defined in Chapter 2 as Equation (2.65).

$$U(x, y, z) = \int_{-\infty}^{\infty} A(k_x, k_y; 0) e^{i(k_x x + k_y y)} dk_x dk_y$$

(4.30)

Hence Equation (4.28) can be represented by the term cylindrical wave expansion. For an easy understanding, the spectral solution in cylindrical coordinates (Equation 4.28) can be compared with the spectral solution in Cartesian coordinates (Equation 4.30). On comparison the following correspondences can be revealed.

$$
egin{align*}
U(x, y, z) &\Rightarrow p(r, \phi, y) \\
A(k_x, k_y; 0) &\Rightarrow P_n(a, k_y) \\
A(k_x, k_y; z) &\Rightarrow P_n(r, k_y) \\
e^{i(k_x z)} &\Rightarrow \frac{H_n^{(1)}(kr)}{\frac{H_n^{(1)}(ka)}{H_n^{(1)}(ka)}} \\
k_x &\Rightarrow n/r \\
k_y &\Rightarrow k_y \\
k_z &\Rightarrow kr where kr = \sqrt{k^2 - k_z^2}
\end{align*}
$$

Thus in view of the fact that $A(k_x, k_y; z)$ is the plane wave (angular) spectrum, $P_n(r, k_y)$ can be called as the helical wave spectrum.

Since the two-dimensional Fourier transform (Equation 4.25) of the left hand side of Equation (4.28) is $P_n(r, k_z)$ then,

$$P_n(r, k_y) = \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} P_n(a, k_y)$$

(4.31)
Equation (4.31) provides the relationship between the helical wave spectrum at different cylindrical surfaces in the same way that $e^{ik_zz}$ provided the relationship between the planar surfaces. The spectral component in Equation (4.28) is given by

$$P_n(a, k_y) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} p(a, \phi, y)e^{-in\phi}e^{-ik_yy}dy$$  \hspace{1cm} (4.32)$$

which is nothing but a Fourier transform relation. The propagation component in Equation (4.28) (Transfer function) is given by

$$T(a, k_r, r, k_r) = \frac{H_n^{(1)}(k_r r)}{H_n^{(1)}(k_r a)}$$  \hspace{1cm} (4.33)$$

where $k_r = \sqrt{k^2 - k_y^2}$ and $k = 2\pi/\lambda$

The propagation of helical wave spectrum is very difficult to visualize both in the axial direction and the radial direction. Some visual ideas on how the propagation of helical wave spectrum can be perceived, is given by Williams [84]. He also provides a discussion on the existence of evanescent wave and the necessary conditions. However these concepts are not important to the reported research work and hence are not mentioned here.

When ‘r’ in Equation (4.28) is kept constant, i.e., the measurement plane (hologram surface) is also a cylinder, then the system is shift invariant. Hence we can use FFT to evaluate Eq. (4.28) and hence fast calculation. Deriving out the analytical expression for the Transfer Function for propagation from cylindrical surface as shown in Equation (4.33), is the most important step in this work.

All the theories explained above are given with greater details by Lebedev [85], Arfken [86]

## 4.2.1 Sampling Conditions

Proper sampling at the object and hologram surface is required for loss free reconstruction. For this, the Nyquist sampling conditions must be satisfied. Consider the transfer function was generated using $N$ samples which runs from $[-N/2...0...N/2]$. According to Nyquist theorem, the discrete transfer function’s rate of change should be less than or equal to $\pi$ at $N/2$. From the analysis of Equation (4.33) one could understand that, the spatial rate of change of $k_r r$ is higher than that of $k_r a$. Hence as long as the sampling condition for $k_r r$ is satisfied, the entire Transfer function also meets the sampling condition approximately. Accordingly the Nyquist sampling condition can be expressed
by the inequality as shown below
\[
\frac{\partial}{\partial n} \left| kr \sqrt{1 - (\lambda k_y)^2} \right|_{n=N/2} \leq \pi \tag{4.34}
\]

From the above inequality with conditions, \( k_y = n \Delta k_y, \Delta k_y = \frac{1}{\Delta L_0} \) and \( k = \frac{2\pi}{\lambda} \), (where \( \Delta L_0 \) is the height of the cylinder) we can obtain
\[
\frac{k\lambda^2 rn}{\Delta L_0^2 \sqrt{1 - \frac{\lambda^2 n^2}{\Delta L_0^2}}} \leq \pi \tag{4.35}
\]
which again reduces to
\[
2nr\lambda \leq \Delta L_0 \sqrt{\frac{\Delta L_0^2}{\lambda^2} - \lambda^2 \left( \frac{N}{2} \right)^2} \tag{4.36}
\]

As \( \Delta L_0^2 \gg \lambda^2 \left( \frac{N}{2} \right)^2 \) and is usually satisfied, a better approximation of the above inequality is
\[
\Delta L_0 \geq \sqrt{N} r \lambda \quad \text{(or)} \quad N \geq \frac{\Delta L_0^2}{r \lambda} \tag{4.37}
\]

Based on this sampling condition (Equation 4.37), the dimensions of the object and hologram were chosen. Accordingly, the object and hologram were assumed to be cylindrical surfaces with radius \( a = 1 \) and \( r = 10 \) respectively. The height of the cylindrical surface was assumed to be \( y = 10 \). To avoid harsh sampling requirements, the wavelength \( \lambda \) was assumed to be large i.e, \( \lambda = 180 \mu m \). When all these dimensions were substituted in the sampling condition, given by Equation (4.37), the required number of samples turned out to be \( N \approx 512 \). Hence the object and the transfer function will be generated as 512×512 matrices for the simulation.

### 4.3 Conclusion

The wave propagation from one cylindrical surface to another based on scalar diffraction theory was discussed. The theory was developed from the solution to wave equation (Equation 4.3) in spectral domain. The spectral propagation formula for cylindrical waves had the same architecture as that of spectral propagation for plane waves. In the case if cylindrical waves, the spectral expansion can be denoted by the term helical wave spectrum, similar to the term plane wave spectrum used for plane wave case. The transfer function was found to be the ratio of Hankel function of first kind (Equation 4.33). The
whole formula can be computed using two FFT operations as in the case of plane waves, and hence is computationally inexpensive.
Chapter 5

Spherical Wave Propagation

5.1 Introduction

The cylindrical CGH reconstructions presented in Chapters 4 could reconstruct an object for $360^\circ$ in the horizontal (azimuth) direction only. The object could not be reconstructed in the polar direction i.e., for top and bottom views of the object was not possible. For three-dimensional information storage and retrieval (of an object) to be complete, recording and reconstruction is to be done on all sides (directions) of an object, i.e., for $360^\circ$ on both azimuth and polar directions. This is achieved by recording and reconstructing on a spherical surface surrounding the object, which is called as spherical holography. However available optical techniques and numerical methods have restricted holography to be done on plane and cylindrical surfaces only. The restriction due to optical techniques arises from achieving coherent illumination of object and reference wave on a spherical surface. One solution is to model the object on the computer and numerically generate the hologram. But still the following issues prevail, i) The unavailability of fast calculation methods or availability of only approximated fast calculation methods for spherical CGH and ii) the need to illuminate the whole spherical surface using a single reference wave front for optical reconstruction. However the availability of optic fibers and developments in multidimensional lasers [87, 88] that include microball lasers or lasing microspheres have provided hope in realizing spherical holography. Motivated by these appealing facts it was decided to make some developments to numerical methods for spherical holography.

Wave propagation in homogeeuous medium for non polarized light is defined by the Rayleigh Sommerfeld diffraction integral [35]. This formula enables us to calculate the optical field (complex amplitude) at an arbitrary point or surface (hologram) knowing the radiating source point or surface (object). This method can be used to generate
a spherical hologram of an object. This method also known as the direct integration method and is slow due to large sampling points in object and hologram. To speed up computation the Rayleigh Sommerfeld integral is expressed as convolution integral and then evaluated using fast Fourier transform (FFT). This speeds up the calculation considerably, however demands the object surface and hologram surface to satisfy shift invariance. Since a spherical holographic surface is not shift invariant with a plane object surface FFT cannot be used for the calculation. Hence the direct integration method has to be used, which consumes $O(N^4)$ or $O(N^3)$ operations for $N$ sampling points.

Fast computation solutions for spherical computer generated hologram employing PSF (convolution method) was proposed by Tachiki et al. [14]. Here the the object and hologram were assumed to be concentric spherical surfaces in order to achieve shift invariance and hence enable fast calculation. However, though the object was assumed to be a concentric spherical surface with the hologram surface, shift invariance does not exist due to unequal sampling points on a spherical grid (ie., the grid points are more crowded at the poles). To facilitate fast calculation using FFT, an approximation to the convolution integral was proposed, which forced the PSF to be spatially invariant. Hence this calculation produced errors which were quantified in the same report.

An accurate method without any approximations for spherical CGH does not exist yet. Such a solution will be possible if we consider this system as a boundary value problem in spherical co-ordinates. Then solve the problem using the boundary conditions, starting from the Helmholtz wave equation. The solution will define the transfer function and the spectral decomposition of the wavefield in the spherical surface. Using the transfer function and the wave spectrum we can develop a spectral propagation formula (for spherical surfaces in spherical coordinates) analogous to the angular spectrum formula (for plane surfaces in cartesian coordinates). This chapter explains such a theoretical development and corresponding solutions. From the solutions a fast computation method for spherical CGH is devised.

The theoretical development in this chapter follows the same procedure presented in Chapter 3. However the treatment varies in the case of spherical co-ordinates because the surface is closed. While in cylindrical co-ordinates it is closed only in the horizontal direction. This fundamental difference also reflects itself in the solution obtained.
5.2 Theoretical background

We will describe points on the sphere by their latitude $\theta \in [-\pi/2, \pi/2]$ and longitude $\phi \in [0, 2\pi]$ as shown in Fig. 5.1. Throughout this work we will be dealing with square-integrable functions that span a space $L^2(S)$ on the unit sphere $S$. Fast computation algorithms take advantage of the cyclic and periodic properties of the transformation kernel for fast calculations. The cyclic and periodic properties exists only if the system is shift invariant between the transformation planes for that operation. Which means that, the object and hologram surface should be shift invariant in order to devise a fast computation scheme. To achieve this the object and hologram surface are chosen to be concentric spherical surfaces, so that they can remain shift invariant in rotation along $\phi$ and $\theta$ directions. But these surfaces are defined on a spherical grid, where the sampling points are more dense at the poles than at the equator. Hence shift invariance is not satisfied. However in the hologram generation process, the hologram and object are band-limited functions on $L^2(S)$ space. The band-limited functions on $L^2(S)$ space has a very useful and important property that any rotated version of a band-limited function is also a band-limited function with the same bandwidth [89, 90]. Thus, they are referred to as having uniform resolution, at all points on the sphere, meaning that they are shift-invariant. The triangular truncation and Gauss-legendre quadrature method that occurs in the transformation operation is responsible for this property (explained in the next section). Hence the system shown in Fig. 5.1 does have shift invariance relationship between object and hologram surfaces and approves the possibility of fast computation formula. With this assurance we proceed to develop the fast calculation method for computer generated spherical holography starting from the basic equations of electromagnetism.

An electromagnetic field is defined by Maxwell’s equations and its propagation by the Helmholtz wave equation. Hence for any particular system the complex amplitude of a propagating wave at any instance of time and anywhere in space, can be found by solving the wave equation, applying its constraints and conditions. Accordingly for the system shown in Fig. 5.1 the solution can be derived starting from the wave equation as follows, The time dependent vector wave equation $u(r, \theta, \phi)$ is expressed as

$$\nabla^2 u - \epsilon \mu \frac{\partial^2 u}{\partial t^2} = 0 \quad (5.1)$$

where $r$ is the radius of the spherical surface of interest and $\theta, \phi$ represent the azimuthal and polar angles in the surface. The Laplacian operator $\nabla^2$ in spherical co-ordinates is defined as

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad (5.2)$$
When the wave is non polarised and the medium of propagation is homogeneous then the vector nature of the function “u” can be discarded. The scalar wave equation given in spherical co-ordinates becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad (5.3)$$

The solution of Eq. (5.3) can be found by separation of variables \([85, 86]\), which can be expressed as shown below

$$u(r, \theta, \phi, t) = R(r)\Theta(\theta)\Phi(\phi)T(t) \quad (5.4)$$

The separable variables obey the following four ordinary differential equations.

$$\frac{d^2 \Phi}{d\phi^2} + m^2 \Phi = 0 \quad (5.5)$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + [n(n + 1) - \frac{m^2}{\sin^2 \theta}]\Theta = 0 \quad (5.6)$$

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + k^2 R - \frac{n(n + 1)}{r^2} R = 0 \quad (5.7)$$

$$\frac{1}{c^2} \frac{d^2 T}{dt^2} + k^2 T = 0 \quad (5.8)$$
The solution to all the above equations are derived in by Arfken [86]. Only the final results are used in this research work. The solution to azimuthal Eq. (5.5) is

$$\Phi(\phi) = \Phi_1 e^{im\phi} + \Phi_2 e^{-im\phi} \quad (5.9)$$

where $m$ must be an integer so that there is continuity and periodicity in $\Phi(\phi)$. $\Phi_1$ and $\Phi_2$ are constants.

The solution to polar Eq. (5.6) is

$$\Theta(\theta) = \Theta_1 P^m_n(\cos\theta) + \Theta_2 Q^m_n(\cos\theta) \quad (5.10)$$

where, $P^m_n$ and $Q^m_n$ are the associated Legendre polynomials of first and second kind respectively and $\Theta_1$ and $\Theta_2$ are constants. $Q^m_n$ is not finite at the poles where $\cos(\theta) = \pm 1$, so this solution is discarded ($\Theta_2 = 0$).

For the radial differential equation Eq. (5.7) the solutions are

$$R(r) = R_1 h^{(1)}_n(kr) + R_2 h^{(2)}_n(kr) \quad (5.11)$$

where, $h^{(1)}_n$ and $h^{(2)}_n$ are the Spherical Hankel functions of the first and second kind respectively. Since we are interested only in outgoing wave, we can neglect the second term ($R_2 = 0$)

The solution to Eq. (5.8) is

$$T(\omega) = T_1 e^{i\omega t} + T_2 e^{-i\omega t} \quad (5.12)$$

Again here we assume $T_2 = 0$ due to the convention of time.

The angle functions in Eq. (5.9) and Eq. (5.10) are conveniently combined into a single function called as spherical harmonic $Y^m_n$ [85, 86] defined by

$$Y^m_n(\theta, \phi) \equiv (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \tilde{P}^m_n(\cos\theta) e^{im\phi} \quad (5.13)$$

where the quantity

$$\tilde{P}^m_n = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P^m_n(\cos\theta) \quad (5.14)$$

is known as the orthonormalized associated Legendre polynomial. The term $(-1)^m$ is called as the Condon-Shortley phase. Hence the spherical harmonics can also be
represented in a short form as shown below (neglecting the Condon-Shortley phase)

$$Y_n^m(\theta, \phi) = P_n^m(\cos \theta)e^{im\phi} \quad (5.15)$$

Combining all the above equations, the travelling wave solution to Eq. (5.3) can be represented as

$$u(r, \theta, \phi, \omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn}(\omega) h_n(kr) Y_n^m(\theta, \phi) \quad (5.16)$$

The radiated field is completely defined when the coefficient $A_{mn}$ is determined. This is achieved by using the orthonormal property of the spherical harmonics. Assume that the wavefield $u(r, \theta, \phi)$ is known on a sphere of radius $r = a$. We also drop the time variable (which is not important) for simplicity. Now multiplying each side of Eq. (5.16) (evaluated at $r = a$) by $Y_n^m(\theta, \phi)^*$ and integrating over the sphere, gives

$$A_{mn} = \frac{1}{h_n(ka)} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} u(a, \theta, \phi) Y_n^m(\theta, \phi)^* \sin(\theta) d\theta d\phi \quad (5.17)$$

where, $d\Omega = \sin(\theta)d\theta d\phi'$, is the solid angle for integrating on a sphere. Inserting the expression for $A_{mn}$ back into Eq. (5.16), we get,

$$u(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_n^m(\theta, \phi) \left( \int_0^{\frac{\pi}{2}} \int_0^{2\pi} u(a, \theta', \phi') Y_n^m(\theta', \phi')^* d\Omega' \right) \frac{h_n(kr)}{h_n(ka)} \quad (5.18)$$

Hence the wavefield at any spherical surface $u(r, \theta, \phi)$ can be calculated knowing the wavefield at $u(a, \theta', \phi')$.

For easy understanding and interpretation of Eq. (5.18), it is compared with the well known angular spectrum of plane waves equation [35] which is given by

$$u(x, y, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(k_x x + k_y y)} dk_x dk_y \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x', y', 0) e^{-i(k_x x' + k_y y')} dx' dy' \right) e^{i k_z z} \quad (5.19)$$

It is well known that in Eq. (5.19) the quantity within square brackets represents the source wavefield decomposed into spectrum of plane waves in $(k_x, k_y)$. The propagation of the spectrum is defined by the quantity $e^{i k_z z}$ which is known as the transfer function. The propagated spectrum is recomposed into the wavefield at destination by the integral over $k_x, k_y$. Thus this equation defines wave propagation from one plane surface $u(x', y', 0)$ to another surface $u(x, y, z)$. When comparing Eq. (5.18) with Eq. (5.19), the following can be deduced.
• Wave field in \((\theta, \phi)\) at a radiating spherical surface of radius \(r = a\) is decomposed into its wave spectra in \((m, n)\) defined by

\[
U_{mn}(a) = \int u(a, \theta, \phi) Y_n^m(\theta, \phi)^* d\Omega \quad (5.20)
\]

• The decomposed wave components (spectra) are expressed by Eq. (5.13) which is composed of a travelling wave component in \(\phi\) and defined by \(e^{im\phi}\) and a standing wave component in \(\theta\) given by \(P_n^m(\cos \theta)\).

• The decomposed wave components in this system can be named as spherical wave components in analogous to plane wave components for planar system. Similarly Eq. (5.20) can be termed as Spherical wave spectrum as analogous to Angular spectrum of plane waves.

• The wavenumbers \(k_x\) and \(k_y\) are imitated by the quantities \(m/a\) and \(n/a\). Hence we can refer to the spherical wave spectrum as a \(k\)-space spectrum due to this analogy.

• Equation (5.20) can be viewed as a forward Fourier transform using \(Y_n^m(\theta, \phi)\) as the basis function. In other words, the spectral space for spherical system is spanned by Spherical harmonic functions \(Y_n^m(\theta, \phi)\). This is also termed as Spherical Harmonic Transform (SHT).

• The propagation of spherical wave spectrum from one spherical surface of radius \(a\) to another of radius \(r\) is given by

\[
U_{mn}(r) = \frac{h_n(kr)}{h_n(ka)} U_{mn}(a) \quad (5.21)
\]

• Hence the quantity \(h_n(kr)/h_n(ka)\) can be referred to the Transfer function (TF) for spherical system, as opposed to the quantity \(e^{ikz}\) for planar system.

• The inverse spherical harmonic transform (ISHT) that recomposes the wave field back from the spectrum is given by

\[
u(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} U_{nm}(r) Y_n^m(\theta, \phi) \quad (5.22)
\]

The transfer function is crucial since it completely defines propagation and hence it is worth discussing some of its properties. During propagation amplitude and phase of the spectral components change with distance as defined by the transfer function. What is most important is the rate of change of phase of the transfer function which determines the sampling requirements. Accordingly the plot shown in Fig. 5.2 reveals that the
Figure 5.2: Plot of phase (in radians) of the transfer function for increasing order (n).

Phase change increases with increasing orders 'n' of the transfer function (The plot was generated for wavelength 100\(\mu\)m, radius 10 and 0.5 cm, and up to 256 orders). Hence sampling requirements will be satisfied if highest order 'n' of the transfer function is sampled according to Nyquist criteria. It can also be understood that, the rate of phase change becomes higher as the distance of propagation increases. This will demand very large sampling and also increase the numerical errors. It is worthy to note here that the spherical Hankel functions are asymptotic in nature. In the far field the spherical Hankel functions can be approximated by their asymptotic expressions which is as shown in Eq. 5.23.

\[
h_n^{(1)}(x) = (-i)^{n+1} \frac{e^{ix}}{ix} \tag{5.23}
\]

This could yield a formula analogous to the farfield Fresnel diffraction formula. However it requires systematic development of theory with proper analysis of approximations and are not discussed in this chapter. But this can be considered as a future work to the proposed method. Thus the devised formula which is analogous to the angular spectrum of plane waves(AS) formula defines wave propagation between spherical surfaces. The numerical procedure to implement the devised formula is discussed in the next chapter.

### 5.3 Conclusion

The wave propagation for spherical systems has been derived from the boundary value solutions to the Helmholtz wave equation in spherical co-ordinates. From the solution the transfer function and wave spectrum were defined. A computation method analogous to the angular spectrum method was devised from these definitions. The only difference is the usage of spherical harmonic transfroms instead of Fourier transform. This formula
could simulate wave propagation from one spherical surface to another and hence could be used to generate spherical holograms.
Chapter 6

Implementation and Results - Cylindrical CGH

6.1 Introduction

This chapter deals with realization and demonstration of the theory explained in Chapter 3. For this, a cylindrical object is modeled and the wave propagation is computed onto the cylindrical hologram. The same computation is done with the reference beam also and thereby the hologram is generated. The generated hologram is tested for simulated reconstruction on the computer. This chapter explains all the procedures and methods adapted for this process, in detail.

As explained earlier, the aim of this work is to generate a cylindrical hologram on the computer by using wave propagation in spectral domain. The necessary theory required for the calculation was discussed in Chapter 3. The object of interest (for which the hologram is to be generated) will usually not be a so called *well behaving function*, but any arbitrary complex amplitude distribution. So the analytical mathematical solutions mentioned in Chapter 3 cannot be used as such. Instead it is required to evaluate the formulae by using numerical methods. Again, this is a spectral propagation formula and hence the transfer function (Equation 4.33) is the important candidate to be computed. Since we evaluate the formula numerically, the transfer function has to be generated as a discrete set. Both these tasks are prone to numerical and discretization errors. Hence the methods and procedures adapted should be properly chosen or designed inorder to minimize the error and at the same time does not compromise on the computation speed. All the computational procedures adapted in this work are explained in this chapter. The results in this chapter are taken from my previous graduation report submitted to Anna university, Chennai, India.
6.2 Hologram Generation

The computational procedure starts with modeling the object and reference on the computer. Then comes the most important task, which is to generate the transfer function and hence calculate the wave propagation from object and reference to the hologram surface. Finally the generated hologram is tested by reconstructing the object from the hologram intensity data. The procedure followed in implementing these are explained in the following sections.

6.2.1 Generation of object and reference

![Figure 6.1: Cylindrical digital hologram recording setup - Schematic](image)

The object and hologram are modeled as concentric cylindrical surfaces with object as the inner surface. The schematic and geometry of the simulation problem are shown in Figures 6.1 and 6.2 respectively. The object and hologram cylindrical surfaces have radii $a$ and $r$ respectively. The height of the cylindrical surfaces is denoted as $y = h$. The object was generated as a 2 dimensional matrix of size $N \times N$. The size of the matrix was chosen such that it satisfies the sampling condition explained in section 4.2.1. The generated object is graphically shown in Figure 6.3. The object is illuminated virtually by a point source from the center which also serves as the reference.
When generating hologram on a computer, the reference source can be placed anywhere. All sources are virtual in calculations, and hence even placing the reference behind the object is possible and will not obstruct any light. This is a very important freedom enjoyed by digital holography. As explained earlier, in this problem the reference is assumed to be a point source at the center of the hologram radiating coherently in all directions. Realizing this kind of coherent reference illumination in real world is very difficult. But simulated reconstruction alone will be enough to verify the correctness of this numerical method. So we use this simple reference illumination for the initial simulation trials.
6.2.2 Generation of transfer function

Once the object and reference are generated, it is required to simulate wave propagation from object and reference. For this the transfer function is needed as explained earlier. The transfer function for this simulation problem was defined in Equation (4.33) which is,

\[ T(a, k_a, r, k_r) = \frac{H^{(1)}_{n}(k_r r)}{H^{(1)}_{n}(k_r a)} \]  

From Equation (6.1) it is clear that the transfer function can be generated only if physical values are assigned to the various parameters. It is done by taking the sampling conditions (Equation 4.37) into consideration. The object and hologram cylindrical surfaces were assigned radius of \( a = 1 \text{ cm} \) and \( r = 10 \text{ cm} \) respectively. The wavelength was assumed to be large \( \lambda = 180 \mu \text{m} \) to avoid very strict sampling requirements. Accordingly, the number to samples required for loss free reconstruction turned out to be \( N = 512 \).

The various parameters required to generate the transfer function are tabulated below.

- \( a = 1 \text{ cm} \);
- \( r = 10 \text{ cm} \);
- \( y = 10 \text{ cm} \);
- \( N = 512 \);
- \( \theta = 2\pi \);
- \( \Delta k_z = \frac{2\pi}{\lambda} \);
- \( k_y = N\Delta k_z \);
- \( k_r = \sqrt{k^2 - k_z^2} \);
- \( k = \frac{2\pi}{\lambda} \);
- \( \lambda = 180 \mu \text{m} \);

The transfer function contains the Hankel function of first kind which belongs to the category of special functions. All these special functions have table of standard values for all possible variable values. Hence the value of the Hankel functions required to calculate the transfer function can be obtained from the already existing standard table. The programing language python is used for generating the transfer function (and also for whole of the computations). The python library has these standard tables and hence the Hankel functions were generated by referring to this table. This also makes the generation of transfer function very fast. The modulus distribution of the generated transfer function is shown graphically in Figure 6.4.
Now everything is well set for calculating the complex amplitude due to the object and reference in the hologram plane. The complex amplitude is calculated according to Equations 4.28 and 4.29 which are once again given below for easy reference.

\[
p(r, \phi, y) = \sum_{n=-\infty}^{\infty} e^{in\phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} P_n(a, k_y)e^{ik_y y} \frac{H_n^{(1)}(k_y r)}{H_n^{(1)}(k_r a)} dk_y
\]  

(6.2)

where,

\[
P_n(a, k_y) = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \int_{-\infty}^{\infty} p(a, \phi, y)e^{-in\phi}e^{-ik_y y} dy
\]  

(6.3)

For easy understanding and proper follow through, the various parameters used in this calculation are described and tabulated as follows.

The above computation procedure can be summarized as follows:

1. The transfer function (TF) was generated according to Equation (4.33).
2. The complex amplitudes of object and reference were generated as 512 × 512 matrices. The generated object is graphically shown in Figure 6.3.
3. The Fourier spectra of object and reference wavefields were computed (using FFT) according to Equation (4.25). This gives the corresponding complex amplitude in spectral domain at the object surface.
4. The calculated spectrum (object and reference) is multiplied with the transfer function which gives the corresponding complex amplitude in spectral domain at the hologram surface.
Table 6.1: Parameters used in the calculation

<table>
<thead>
<tr>
<th>Wave field in Object surface</th>
<th>General</th>
<th>Object</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular spectrum in Object surface</td>
<td>$P_n(a, k_y)$</td>
<td>$A_o(a, n, k_y)$</td>
<td>$A_r(a, n, k_y)$</td>
</tr>
<tr>
<td>Angular spectrum in Hologram surface</td>
<td>$P_n(r, k_y)$</td>
<td>$A_o(r, n, k_y)$</td>
<td>$A_r(r, n, k_y)$</td>
</tr>
<tr>
<td>Wave field in Hologram surface</td>
<td>$p(r, \phi, y)$</td>
<td>$U_o(r, \phi, y)$</td>
<td>$U_r(r, \phi, y)$</td>
</tr>
<tr>
<td>The Transfer Function</td>
<td>$T(a, k_{a}, r, k_{r})$</td>
<td>$T(a, k_{a}, r, k_{r})$</td>
<td>$T(a, k_{a}, r, k_{r})$</td>
</tr>
</tbody>
</table>

Table 6.2: The step by step computation procedure

<table>
<thead>
<tr>
<th>STEP</th>
<th>Action</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Generation of object - $U_o(a, \phi, y)$</td>
<td>$U_o(a, \phi, y)$</td>
</tr>
<tr>
<td>2</td>
<td>Generation of reference - $U_r(a, \phi, y)$</td>
<td>$U_r(a, \phi, y)$</td>
</tr>
<tr>
<td>3</td>
<td>Computation of angular spectrum of Object - $A_o(a, n, k_y)$</td>
<td>$FFT[U_o(a, n, k_y)]$</td>
</tr>
<tr>
<td>4</td>
<td>Computation of angular spectrum of Reference - $A_r(a, n, k_y)$</td>
<td>$FFT[U_r(a, n, k_y)]$</td>
</tr>
<tr>
<td>5</td>
<td>Generation of Transfer function (TF) - $T(a, k_{a}, r, k_{r})$</td>
<td>$T(a, k_{a}, r, k_{r})$</td>
</tr>
<tr>
<td>6</td>
<td>Computation of AS of Object at hologram plane - $A_o(r, n, k_y)$</td>
<td>$T \times A_o(a, n, k_y)$</td>
</tr>
<tr>
<td>7</td>
<td>Computation of AS of Reference at hologram plane - $A_r(r, n, k_y)$</td>
<td>$T \times A_r(a, n, k_y)$</td>
</tr>
<tr>
<td>8</td>
<td>Computation of Object field at hologram plane - $U_o(r, n, k_y)$</td>
<td>$IFT[A_o(r, n, k_y)]$</td>
</tr>
<tr>
<td>9</td>
<td>Computation of Reference field at hologram plane - $U_r(r, n, k_y)$</td>
<td>$IFT[A_r(r, n, k_y)]$</td>
</tr>
<tr>
<td>10</td>
<td>Computation of Hologram</td>
<td>$</td>
</tr>
</tbody>
</table>
5. The complex amplitude in spectral domain is inverse Fourier transformed according to Equation (4.26) to get the complex amplitude in real space at the hologram surface.

6. The complex amplitudes due to object and reference are added (superposed) at the hologram surface and their resulting intensity is calculated.

In the language of computer programming, the computation procedure can be written as

\[
Hologram = |IFFT[FFT(Object) \times TF] + IFFT[FFT(Reference) \times TF]|^2
\]

where, \( Hologram \) is the resulting 2D image(matrix) that holds the intensity pattern of hologram, \( TF \) is the transfer function and \( FFT \) and \( IFFT \) are the forward and inverse discrete fast Fourier transforms respectively. The calculated intensity pattern is the hologram and is graphically shown in Figure 6.5.

![Figure 6.5: Computer generated Hologram for the object shown in Figure 6.3](image-url)
6.3 Simulated Reconstruction

To test whether the generated hologram could reconstruct the image of the object, simulated reconstructions were done. The same numerical method that was used for generating hologram, is used to reconstruct the hologram. The reconstruction surface was a cylindrical surface with radius \( a = 1 \text{ cm} \). In other words, the hologram was reconstructed on the original object surface. Hence we expected the whole object to get sharply reconstructed. As explained in the general theory of holography, the real image could be reconstructed only if illuminated with the conjugate of the reference. So the reconstruction is simulated such that the conjugate of the reference is used to illuminate the hologram. The numerical computation of the reconstruction can be expressed as follows,

\[
Reconstruction = |IFFT[FFT(Hologram \times \text{Conjugate}[Reference]) \times TF]|^2
\]

where, \( Reconstruction \) is the 2D image(matrix) that holds the reconstructed intensity distribution, \( TF \) is the transfer function and \( FFT \) and \( IFFT \) are the forward and inverse discrete fast Fourier transforms respectively. The reconstructed image pattern is shown in Figure 6.6 which exactly resembles the original object chosen.

6.4 Testing the Hologram

The simulated reconstruction mentioned in the previous section was done using the same numerical method that was used for the generation process. This does not confirm
the accuracy of the computation method. Hence it was decided to subject the generated hologram for further tests. For this, three more reconstruction arrangements were considered which are explained below.

6.4.1 Reconstruction in Plane Surface

Instead of reconstructing on the cylindrical surface, it was decided to reconstruct the cylindrical object on a plane surface. By this we expect only the portion of the cylindrical object that coincides with the plane surface to be reconstructed and the other portions blurred. The plane surfaces were defined at distances $z = 1$ and $z = -1$ from the center of the object surface. The schematic and geometry of this arrangement are shown in Figures 6.7 and 6.8. The width and height of the reconstruction plane were set to be $3\, cm$ and $10\, cm$ respectively. Each reconstruction plane coincided with the original cylindrical object only at one position, i.e., at $\theta = 0^\circ$ and at $\theta = 180^\circ$. Hence during reconstruction, only these two vicinities (of the object) were expected to be reconstructed in focus and other areas unfocused. But, these two reconstruction planes did not maintain any shift invariance relation to the hologram surface and hence FFT could not be used for reconstruction. Hence the reconstruction was done using direct integration according to Equation (6.4) shown below

$$U(a, \phi, z) = \int \int \frac{U(r, \theta, z)\exp(ikr)}{r}d\phi dz$$  \hspace{1cm} (6.4)

Figure 6.7: Reconstruction of hologram in plane surface- Schematic
The reconstructed images for planes $z = 1$ and $z = -1$ are shown in Figure 6.9 and Figure 6.10 respectively. As seen from the Figure 6.9 only the letter ‘P’ that lies in the vicinity of $\theta = 0^\circ$ is in focus. While Figure 6.10 shows the letter ‘O’ which is in the vicinity of $\theta = 180^\circ$ reconstructed sharply. Hence the cylindrical hologram could reconstruct the stored data properly in any surface.
6.4.2 Reconstruction for Variable Viewing Angles

The most important and distinct property of a cylindrical hologram is its ability to reconstruct in all directions i.e., 360°. So when viewing from a particular direction, only the portion of the hologram that falls within the viewing angle is visible and the others hidden. An attempt was made to demonstrate this capability of the cylindrical hologram. This attempt could also serve as another verification for the accuracy of the computation method. Accordingly the model of the reconstruction problem was designed as shown in Figure 6.11.

Three cases were considered, one for a viewing angle of 120° i.e., (−60° to 60°) and another for 90° (90° to 180°) and the other for 45° (−180° to −135°). The direct integration formula, shown in Equation (6.4) was used to simulate reconstruction. The reconstructed results for viewing angle −60° to 60°, 90° to 180° and −180° to −135° are shown in Figure 6.12, Figure 6.13 and Figure 6.14 respectively. The letters ‘APA’ that fall in the vicinity of −60° to 60° are alone reconstructed in Figure 6.12. While the letter ‘C’ and half of the letter ‘O’, that fall in the vicinity of −180° to −90° are reconstructed in Figure 6.13. Similarly the view angle of 45° reconstructs only half of the letter ‘O’ and half of ‘R’ that lie within −180° to −135° as shown in Figure 6.14. Hence the hologram was able to reconstruct the stored information properly.
Figure 6.11: Reconstruction of hologram for variable view angles - Schematic

Figure 6.12: Reconstruction of cylindrical hologram in $120^\circ (-60^\circ$ to $60^\circ$) angle

Figure 6.14: Reconstruction of cylindrical hologram in $45^\circ (-180^\circ$ to $-135^\circ$) angle
6.5 Multiple surface objects

The object was chosen to be a single cylindrical surface in the simulation problem explained in the previous section. This was to keep the problem less complicated and for ease of computation during the initial simulation trials. But holography is meant to record and reproduce information in 3-dimensions (not information only from a single surface). In other words, hologram can reconstruct information from multiple surfaces at different depths. Hence it was planned to generate the hologram of a more complex object with two cylindrical surfaces one inside the other. The schematic model and geometry of the problem are shown in Figures 6.15 and 6.16 respectively.

The two cylindrical object surfaces had radii $r_1 = 1 \text{ cm}$ and $r_2 = 0.5 \text{ cm}$ respectively from the center. The cylindrical hologram surface had a radius of $10\text{ cm}$ from the center.
The outer cylindrical object surface is denoted as *Object-1* and the inner as *Object-2* and are graphically shown in Figures 6.17 and 6.18 respectively. The reference wavefront is assumed to be the same as in the previous case. The simulation problem consists of generating a single hologram that has stored both the objects and then reconstructing the objects individually from the hologram. This is explained in the next section.

Figure 6.16: Two surface object - Geometry

Figure 6.17: Object - 1
6.5.1 Computational Procedure

First the wavefield due to *Object-1* was calculated in the hologram plane and then for *Object-2*. The same computation method explained in the previous sections was used. So now we have two different wavefields arriving at the hologram surface. These two complex wavefields along with the reference interfere at the hologram surface to give fringe pattern. This is simulated by adding all the three complex wavefields and calculating the modulus. The calculated resultant fringe pattern at the hologram surface is shown in Figure 6.19. Now, this fringe pattern (hologram) has the complete information about both the objects. Hence both the objects should be reconstructed individually at their respective locations from this single hologram. The same reconstruction procedure adapted in the previous simulations was used here except for the fact that, when reconstructing *Object -1*, the corresponding value \( r_1 = 1 \, \text{cm} \) was used. Similarly when reconstructing *Object -2* the value \( r_2 = 0.5 \, \text{cm} \) was used. The corresponding reconstructions for *Object-1* and *Object -2* are shown in Figures 6.20 and 6.21 respectively. It is seen from the figures that the reconstruction of *Object-1* has the unfocused image of *Object-2* and vice-versa.
**Figure 6.19:** Resultant hologram of Objects shown in Figures 6.17 and 6.18

**Figure 6.20:** Object-1: Reconstructed without segmentation
Though this result is expected and is theoretically correct, this is clearly an unwanted effect. In order to rectify this problem the segmentation algorithm described by McElhinney et al. [91] was applied during reconstruction. This algorithm uses the fact that the focused image is more sharp and intense compared to the unfocused one. Accordingly, it calculated variance in the reconstructed images and created a variance map of each pixels. A predefined threshold value was assigned to the variance map and hence a corresponding filter was created. This filter removed the unfocused image from the reconstruction. This method is called as segmentation. The segmented reconstructions for Object-1 and Object-2 are shown in Figures 6.22 and 6.23 respectively. However the figures show that segmentation process affects the focused image as well and hence the expected quality in reconstruction could not be achieved. If the threshold value is set very low so that the segmentation process does not affect the focused image, then the unfocused image is not removed completely. So a kind of compromise is to be setup inorder to get an optimum reconstruction. The reason for this is that, as seen from the figures 6.17 and 6.18, both the objects are exactly one behind the other and overlap each other, which usually does not occur in practical situations. In real world light only propagates from the outermost surface which may be either cylindrical or uneven and non cylindrical. So to achieve proper segmentation, it was planned to model another multiple surface object where the objects do not overlap each other. This is explained in the next section.
6.6 Three Dimensional Object

It was planned to investigate the 3D recording and reconstruction abilities of this numerical method. For this a 3D object was modeled in the computer and then sliced into cylindrical surfaces. Then wave propagation was simulated from each surface and hence hologram was generated. Then from this hologram, the 3D object was reconstructed back. The methods and techniques involved in this simulation process are explained in this section.
6.6.1 Generation of Object

An attempt was made to verify 3-dimensional recording and reconstruction capabilities of a cylindrical hologram. For that, the 3D object was modeled such that it projected in all the directions without any radial symmetry. The object consisted of a triangle and circle with their faces perpendicular to each other. The generated object accordingly is shown in Figure 4.36. Figure 6.24(a) is the view of the object at 0°, and Figure 6.24(j) shows the same object rotated through 90° towards the left hand side. All the other figures, (Figure 6.24(b) to Figure 6.24(i)) represent the intermediate view points from 0° to 90° respectively. The object was generated using the visualization tool kit package which is a graphics processing pipeline. The size of the object used was 1 cm.
Figure 6.24: Different view angles of the generated 3D object
6.6.2 Slicing the Object

The numerical method proposed in this research work is capable of simulating wave propagation from one cylindrical surface to another. Hence inorder to use this method the 3D object should be sliced into concentric cylindrical surfaces (Figure 6.25). Then the wave propagation from each cylindrical surface to the hologram plane is calculated. Accordingly this 3D object was sliced into 64 cylindrical surfaces. The visualization toolkit package (which is a graphics processing pipeline) had the tools necessary to do the slicing operation. After slicing each cylindrical surface had the corresponding pixel data as point sources on their surface. This is shown in Figure 4.38. We have 64 images (2-dimensional) now each representing a cylindrical surface. Then the wavefield due to each cylindrical surface was calculated on the hologram plane individually. The reference source was assumed to be a point source at the center. The reference wavefield and the individual wavefields of all the surfaces were added up in the hologram plane to generate the hologram. The generated hologram is shown graphically in Figure 6.27.

![Figure 6.25: Slicing the 3D object- Schematic](image)
Figure 6.26: 3D object as point sources after slicing
6.6.3 Reconstruction of 3D Object

The hologram is nothing but a 2-dimensional image as shown in Figure 6.27. From this 2D image, it was tried to reconstruct the whole 3D object. In order to achieve this, first the individual cylindrical surfaces were reconstructed one by one. As seen from the earlier sections, the unfocused image points also were reconstructed. The segmentation algorithm was used to remove the unfocused image points in each surface. So now we have a set of 64 2D images each representing a cylindrical surface. Now, again the visualization toolkit package was used to combine all the individual surfaces and regenerate the original 3D object. The reconstructed objects that correspond to the original object are shown in Figures 6.40 respectively. These reconstructions (Figure 6.28(a) through Figure 6.28(j)) agree with the original object chosen. Hence the capability of this numerical method to record and reconstruct 3D object is verified.
Figure 6.28: Reconstructed 3D object
However as seen from Figure 4.40, the base portion of the triangle and some horizontal portion of the circle are missing in the reconstruction. This can be explained as follows. During slicing, the cylindrical surfaces intersect with the axial object points such that they lie one behind the other. The missing horizontal portions also correspond to the same and hence occur as point sources one behind the other in a straight line. Hence they are not fully reconstructed. However if a pyramid shaped structure was chosen instead of the 2-dimensional triangle structure, or a sphere instead of the 2-dimensional circle, then during slicing, the cylindrical surface intersects the point sources such that they are not one behind the other. This reconstruction will not have any missing points.

6.7 Optical Reconstruction

Apart from the simulated recording and reconstruction, it was also planned to do some optical reconstructions. The hologram was generated on the computer and then printed on a film using a fringe printer. Then the hologram was optically illuminated using a laser light and the reconstructed real image was observed. This section explains this process in detail.

He-Ne laser was used for reconstruction and hence for simulation, the wavelength chosen was $\lambda = 633\, nm$. The object and hologram were of height $6.048\, cm$. The radii of object and hologram cylindrical surfaces were $1\, cm$ and $5\, cm$ respectively. On applying these parameter values to the sampling condition (Equation 4.37) the required number of samples turned out to be $46080 \times 8640$ pixels. So the object was generated as a $46080 \times 8640$ matrix which is shown graphically in Figure 6.29.

![CORE OPTICS](image)

**Figure 6.29:** Generated cylindrical object for optical reconstruction
All the previous simulation problems had the reference to be defined as a point source at the center. But as explained earlier it is impossible to realise such a reference illumination in the laboratory. Hence a different setup was sought for reconstruction which is shown in Figure 6.30. Accordingly a collimated laser beam was spread radially using a conical mirror. The radially propagating beam was incident on the cylindrical holographic surface at an angle of 20°. This was oblique incidence of reference and is very similar to offaxis holographic recording setup. The reference was generated using the following expression.

$$U_r(\phi, y) = E_r e^{\frac{2\pi i}{\lambda} y \sin \theta}$$  \hspace{1cm} (6.5)

where the angle of incidence was $\theta = 20^\circ$. The object wave was simulated using the spectral propagation formula as in the case of earlier simulations. Once the reference and object wavefield were calculated in the hologram plane, the interference pattern was calculated and the hologram was generated. For this, it was required to process $46080 \times 8640 = 398131200$ complex numbers. Generating such a hologram is computationally expensive. It took 4 hrs and 45 min to generate and save the results on 2.6 GHz computer with 12 GB of RAM memory.
Then, the generated hologram was to be transferred on to a film for optical reconstruction. For this a fringe printing setup proposed by Yamaguchi [13] was used. The optical setup of the fringe printer is shown in Figure 6.31. The LCoS SLM displayed the fringe pattern, which was demagnified by a telecentric lens system and focused on to the film. The film was placed on a X-Y translational stage which was at the focal point of a telecentric lens system. The film used for recording purpose was VRP-M manufactured by Slavich. The very big hologram of size $46080 \times 8640$ was cropped in to small parts according to the size of the LCoS SLM. The cropped hologram fringe pattern was fed to the SLM which was demagnified and displayed on the film kept in the translational stage. After one recording the film was moved in the horizontal or vertical direction and the next part of the fringe was recorded. Thus the whole fringe pattern was transferred onto the film with such multiplexing technique. The generated hologram pattern was printed at a resolution of 7 $\mu$m. Therefore a hologram of size 5 cm radius and 6.048 cm height was obtained.

The hologram was printed in three separate pieces. Printing the generated hologram is a very challenging task, because each complex number takes 256 bytes of memory space. Hence one hologram occupies a total of $398131200 \times 256 = 101921587200$ (approx 100 GB) of data. Transferring such large data digitally on to a holographic film takes a lot of time. It took 33 hrs to print the hologram.
Then the printed holographic film was folded into a cylinder and illuminated according to the setup shown in Figure 6.30. The conical mirror used in the reconstruction setup was a custom made one with aluminum foil. When illuminated with He-Ne laser, the mirror scattered light very badly and hence the reconstructed object could not be recognized. So it could not be verified whether the hologram had recorded the information properly. However another reconstruction setup shown in Figure 6.32 was used. Accordingly, the three pieces of the hologram were illuminated separately using a collimated beam. The reconstructed images were observed on a flat screen. The observed reconstructions are shown in Figure 6.33. The objects could be recognized from the reconstructions. However few other problems with the reconstruction setup generated a lot of background noise. But still it proves that the proposed numerical method stored the information properly.

![Figure 6.32: Modified setup used for reconstruction](Image)

**Computer Generated Object**

![CORE OPTICS](Image)

**Optical Reconstructions (real image)**

![Figure 6.33: Optical reconstruction - real image](Image)
6.8 Conclusion

The simulation results presented in this chapter proved that this numerical method can be used to generate cylindrical holograms. Verification experiments confirmed that the object information was properly stored in the hologram. Recording and reconstruction of 3D object was also succesful. Experimental verification was done by optical reconstruction where the reconstructed image resembled the object. Hence it could be concluded that this proposed method was successful in generation and reconstruction of cylindrical hologram digitally.
Chapter 7

Implementation and Results - Spherical CGH

7.1 Introduction

The theoretical development of the computation method for spherical computer generated hologram was explained in the chapter 5. It is now required to test and verify the computation method in context to holography. This chapter presents the simulation experiments to test the computation method along with the produced results. For this, a spherical object is modeled and the wave propagation is computed from object to the spherical hologram surface. The transfer function and wave spectrum defined in the previous chapter are used for this purpose. The same computation is done for the reference beam also and thereby the hologram is generated. The generated hologram is tested for simulated reconstruction on the computer. This chapter explains all the procedures and methods adapted for this process, in detail.

The theory discussed in chapter 5 confirmed the availability of shift invariance in the spherical system and hence fast computation is possible. However to realise fast computation, the numerical method of choice to compute the spherical harmonic transform (SHT) becomes very important. There are a lot of numerical methods proposed for SHT calculation (like for FFT) and the one suitable for this research was carefully chosen. The properties and calculation procedure of this numerical methods are explained in this chapter. A comparision of the simulation results and computation time for the proposed method with the other existing methods is also presented.
7.2 Numerical computation

The numerical computation of angular spectrum (AS) method heavily depends on the FFT operations for which a lot of tools and methods are available. But the numerical computation of the proposed method heavily depends on the SHT operations. Fast computation was guaranteed from the theory and from the geometry of the system. Now a numerical procedure that takes advantage of this is required. Fortunately lot of fast computation numerical methods have been reported for SHT and this research work could make use of it. Since FFT and numerical computation of AS method are well understood, this section intends to introduce SHT and numerical computation of proposed method in close analogy to the former.

Continuing with the comparision from previous chapter, the numerical computation of wave propagation for planar and spherical systems according to Eq. (5.18) and Eq. (5.19) can be represented respectively as

\[ u(r, \theta, \phi) = \text{ISHT} [\text{SHT} (u(a, \theta, \phi)) \times T F_s] \]  
\[ u(x, y, z) = \text{IFFT} [\text{FFT} (u(x, y, 0)) \times T F_c] \]  

where, \( \text{FFT} [...] \) and \( \text{IFFT} [...] \) denote the foward and inverse fast Fourier transform operations, while \( \text{SHT} [...] \) and \( \text{ISHT} [...] \) denote the forward and inverse spherical harmonic transform operations. \( T F_s \) and \( T F_c \) are the transfer functions for spherical and planar wave propagations as defined by Eq. (5.18) and Eq. (5.19) respectively. Comparison reveals that the computation method (for spherical system) is analogous to the angular spectrum of plane waves method (for plane system) except the fact that, the Fourier Transform is replaced by the Spherical harmonic transform. Hence evaluation of SHT becomes the key, the others being only basic mathematical evaluations. SHT operations are fundamentally different from the FFT operations. In the FFT operation the transform is done with sinusoid and its higher harmonics as the basis function, where as in the case of SHT the spherical harmonics defined by Equation 5.13 is the basis. This also implies that since the transformation kernel for the FFT is a sinusoid it is an operation that is defined on a circle. But the spherical harmonics are functions on a sphere and hence the transformation is an operation defined on a sphere. Spherical harmonic transform have been studied extensively and fast computation algorithms and optimisation methods have been proposed. A method that requires only \( O(N^2 \log N) \) operations for \( N \) grid points as opposed to the standard \( N^3 \) operations was proposed by Chien et al. [89]. They imposed truncations on the spectral components and used fast multipole method and fast Fourier transform for evaluation. Their method is called as “spectral
“truncation method” and the errors due to truncation were well within acceptable limits. Later on Healey et al. [92] could achieve the same using \( O(N(\log N))^2 \) operations. They took advantage of the recursive properties of associated Legendre polynomial for fast computation. This method was found to be more efficient and hence was chosen for numerical evaluation the Spherical harmonic transforms in this work. A brief outline of the numerical evaluation is presented. For more details please refer to Healey [92] and Driscoll [90].

Though FFT and SHT are fundamentally different, they both are variable separable. Which means, a 2D FFT is computed by separating the transformation kernel into its variables, and evaluating it as 1D column transform followed by a 1D row transform. Similarly the spherical harmonic transform kernel \( Y^m_n \) given by Eq. (5.15) is also variable separable and can be separated into \( \phi \) component and \( \theta \) component. Then it is evaluated as a 1D transform along the \( \phi \) direction followed by another 1D transform along the \( \theta \) direction. Accordingly the transformation given by Eq. (5.20) can be represented as shown below:

\[
U_{mn}(r) = \int_{-\pi/2}^{\pi/2} \left( \int_{-\pi}^{\pi} u(r, \theta, \phi)e^{im\phi}d\phi \right) \bar{P}^m_n(cos \theta)d\theta 
\]  (7.3)

First, the quantity within the round brackets alone is to be computed which is nothing but a Fourier transform operation. The Fourier coefficients \( U^m(\theta) \) are evaluated for \( m = -N, ..., N \) as shown below:

\[
U^m(\theta) = \int_{-\pi}^{\pi} u(r, \theta, \phi)e^{im\phi}d\theta 
\]  (7.4)

\[
= \frac{1}{I} \sum_{i=1}^{I} u(r, \theta, \phi_i)e^{im\phi_i} 
\]  (7.5)

where \( \phi_i = 2\pi i/I \) for \( i = 1, ..., I \). The equispaced longitudes \( \phi_i \) enables the use of fast Fourier transform.

Second, the Legendre transform of the Fourier coefficients \( U^m(\theta) \) is to be evaluated for \( |m| \leq n \leq N \). This is done using the Gaussian-Legendre quadrature as shown below,

\[
U_{nm} = \int_{-\pi/2}^{\pi/2} U^m_{m}(\theta) \bar{P}^m_n(cos \theta)\sin \theta d\theta 
\]  (7.6)

\[
= \sum_{j=|m|}^{N} U^m_{m}(\theta_j) \bar{P}^m_n(cos \theta_j)w_j 
\]  (7.7)
where \( \theta_j \) and \( w_j \) are respectively the Gauss nodes and weights and are calculated using the Fourier-Newton method as described by Swarztrauber [93]. The Gauss-Legendre quadrature replaces the integral by the sum. The fact that the summation runs only from \( |m| \) to \( N \) is referred to as the triangular truncation. The use of Gauss-Legendre quadrature method redistributes \( \theta \) into Gaussian nodes \( \theta_j \). This along with the triangular truncation are responsible for uniform resolution on the latitudinal points. Again the triangular truncation along with the recurrence property of Legendre polynomial helps to achieve fast computation.

In the similar way the inverse spherical harmonic transform can be represented as

\[
 u(\theta, \phi) = \sum_{m=-N}^{N} \left( \sum_{n=|m|}^{N} U_{nm} P_n^m(\cos \theta) \right) e^{im\phi}
\]  

(7.8)

The inverse transform also follows the same procedure and is computed in two steps but in the reverse order (ie, legendre transform first and Fourier transform next) as shown below

\[
 U_m(\theta) = \sum_{n=|m|}^{N} U_{nm} \tilde{P}_n^m(\cos \theta)
\]  

(7.9)

\[
 u(\theta, \phi) = \sum_{m=-N}^{N} U_m(\theta) e^{im\phi}
\]  

(7.10)

Thus using this numerical procedure fast computation of wave propagation in spherical computer generated holograms is achieved, which uses only \( O(N(\log N)^2 \) operations for the SHT computation.

There are a lot of software tools available on the internet to do the SHT operation. Most of them are tuned and dedicated for a geophysical process which requires only real SHT but holography requires complex SHT. However the package SHTools by Mark Wieczorek [94] could do a complex SHT operation in Fortran language. Though not tested by us this is the best one available that suits the work related to this paper. Hence we recommend using this if it is required to quickly reproduce the work in this paper. The next section describes the testing and verification of this numerical procedure using simulations.

### 7.3 Simulation results

The system considered for simulation experiments is shown in Fig. 5.1. The object \( (O(a, \theta, \phi)) \) is a spherical surface of radius 1cm and the hologram \( (H(r, \theta, \phi)) \) is another concentric
spherical surface of radius 10 cm. The reference is considered to be a virtual source emitting spherical waves from the center, i.e., the wavefield due to reference has the same phase and amplitude throughout the hologram plane. This is similar to using a plane reference wave with normal incidence in plane holography.

### 7.3.1 Verification through comparison

Since this is the first occurrence of such a formula in computer generated holography, it is first required to test it to obey the basic diffraction laws. For this, the proposed method by expecting it to reproduce the already known diffraction results. To achieve this, the proposed method is subjected to generate already reported diffraction patterns for spherical surfaces. For this, the diffraction pattern reported by Tachiki et al. [14] for spherical surfaces is used as reference. Accordingly, a simple object was chosen which is a spherical surface with two irradiating points at $\phi = -\pi/2$ and $\phi = \pi/2$, as shown in Fig. 7.1. The object and hologram are composed of 256 pixels in the longitudinal (north-south) direction and 512 pixels in the latitudinal (east-west) direction. The wavelength was chosen to be $\lambda = 100 \mu m$, in order to reduce sampling requirements and visibility of fringes. The procedure for numerical generation of hologram using the proposed method is expressed in an abstract form as shown below.

\[ \text{AmplitudeHologram} = |(\text{ISHT} [\text{SHT} (\text{Object}) \times TF]) + \text{ISHT} [\text{SHT} (\text{Reference}) \times TF]|^2 \]

Then a hologram for the same object was simulated using the well known direct integration formula defined in Eq. (7.11).

\[ H(r, \theta, \phi) = \iint \frac{O(\theta', \phi') \exp(ikL)}{L} d\theta' d\phi' \]

\[ L = \sqrt{r^2 + a^2 - 2ra \sin(\theta) \sin(\theta') + \cos(\theta) \cos(\theta') \cos(\phi - \phi')} \]

\[ \text{Hologram} = |H_{\text{object}}(r, \theta, \phi) + H_{\text{reference}}(r, \theta, \phi)|^2 \]

The simulation results are shown in Fig. 7.2. The pattern generated by the proposed method matches with the one generated by direct integration method. However, the distribution of brightness and contrast across the pattern is constant for the direct integration method while it decreases gradually from the center for the proposed method. This inconsistency can be explained as follows. The direct integration formula Eq. (7.11) is the Rayleigh-Sommerfeld diffraction formula [14] without the obliquity factor. The obliquity factor is the cosine of the angle between the normal of the radiating surface to the direction of the observation point. This is responsible for the distribution of light
intensity based on the angle (i.e., more bright at the center and gradually decreases outwards and no radiation backwards). However, the spectral method which is the solution to the boundary value problem of the wave equation, incorporates the obliquity factor and hence the brightness and contrast varies radially. Moreover, the obliquity factor does not alter the phase of the traveling wave which in turn does not affect interference pattern and hence guarantees a fair comparison.

### 7.3.2 Verification for diffraction properties

Then the proposed method is tested to see whether it obeys the fundamental laws of diffraction and interference. In other words, it is to verify that it has the same qualities and produces the same results as other diffraction theories. For this two simulation experiments for qualitative analysis was performed. First it was intended to analyze the change in interference pattern with change in wavelength which is a fundamental law. Accordingly for the object shown in Fig. 7.1, the hologram was computed for wavelengths varying as a) 150 µm, b) 200 µm, c) 250 µm, d) 300 µm, e) 350 µm and
f) 400µm respectively. The corresponding holograms generated are shown in Fig. 7.3. As expected the fringe density decreases with increase in wavelength.

![Figure 7.3: Computed hologram(intensity) for wavelength a)150µm, b)200µm, c)250µm, d)300µm, e)350µm, f)400µm.](image)

Second another fundamental law which is the change in interference pattern with the change in distance between the coherent sources was verified. This also corresponds to the youngs double slit experiment. Accordingly, the hologram is computed for varying position of the pair of the point sources on the spherical surface. This setup is similar to the young double slit experiment. The positions of point sources were set to be a) \(\phi = -\pi/6, \phi = \pi/6\), b) \(\phi = -\pi/8, \phi = \pi/8\), c) \(\phi = -\pi/16, \phi = \pi/16\) and d) \(\phi = -\pi/32, \phi = \pi/32\) as shown in Fig. 7.4(a)-(d) respectively. The corresponding computed hologram pattern are shown in Fig. 7.4(e)-(h). As expected the fringe density decreases with decrease in distance between the point sources.

Since the diffraction formula agrees well with the fundamental laws of diffraction it is confirmed that it behaves like the any other diffraction formula and can be used to simulate wave propagation. Moreover the above mentioned results also reveals that the wavefield on the spherical surface was computed correctly and as expected.
7.3.3 Hologram generation

Since the theory is developed in context to computer generated holography, it is mandatory to verify its applicability to the same. For this it was decided to perform spherical hologram generation and then reconstruction from the hologram on the computer using the proposed method. The object was assumed to be a single spherical surface with some images inscribed on it. The spherical object for which the hologram is to be made is shown in Fig. 7.5. The object and hologram were composed of $256 \times 512$ pixels. The wavelength for simulation was chosen to be $\lambda = 30 \mu m$. Again here the wavelength was assumed to be large inorder to reduce the sampling requirements. Now, using the developed formula, wave propagation was simulated from the object surface to the hologram surface. Since the reference was assumed to be a spherical wave emanating from the
center, it contains the same phase and amplitude at the hologram surface. So we have the complex amplitudes of the object and reference as a matrix of complex numbers at the hologram surface. By adding these two complex amplitudes and calculating the intensity will produce the hologram. The generated hologram is shown in Fig. 7.6.

![Figure 7.5: Object.](image1)

![Figure 7.6: Hologram(intensity).](image2)

From this hologram the object was reconstructed back onto the original spherical surface using Eq. 5.18. Reconstruction with the original reference will produce only a virtual image at the location of the object. In order to obtain a real reconstructed image at the original object location, the hologram should be illuminated (or reconstructed) using the conjugate of the reference. This means that we are attempting to reconstruct a real image on the spherical surface where the object was earlier located. The conjugate of the reference was produced by taking the complex conjugate of the reference wavefield matrix. Accordingly the numerical procedure of the for reconstruction is expressed in an abstract form as shown below.
Reconstruction = |\text{ISHT}[\text{SHT}(\text{Hologram} \times \text{Conjugate}[\text{Reference}]) \times TF]|^2

\begin{table}[h]
\begin{tabular}{|c|c|}
\hline
\text{Computation Method} & \text{Time (seconds)} \\
\hline
Direct Integration & 3730.63 \\
Convolution & 0.0573 \\
Spherical spectrum & 0.0296 \\
\hline
\end{tabular}
\end{table}

The reconstructed real image is shown in Fig. 7.7. The reconstruction matches exactly with the object chosen. The reconstruction is crisp and is also free from any noise. As mentioned earlier, the object and hologram are square integrable band limited functions on a closed surface. Hence a rotated(shifted in theta or phi) version of the object or hologram will produce a rotated version of the reconstruction. The wave propagation calculation requires $O(N \log N)^2$ operations for $N$ sampling points and hence it is a fast computation formula. The calculations were executed using a scripted language-python in a Dell Precision T7400 machine with 12 GB of RAM memory. A comparison of calculation time for the direct integration, convolution and spectral methods is shown in Table 7.1. As seen from the table the proposed method took the least time for calculation and hence is the fastest.
7.3.4 Three dimensional reconstructions

The very important feature of hologram is the ability for three dimensional reconstruction. In this research an attempt was made to demonstrate 3D reconstructions using the proposed formula using spherical holograms. We model the object as considering two spherical surfaces containing different images as shown in Figure 7.8 and Figure 7.9. The object spherical surfaces were assumed to have radii of 1cm and 5cm respectively. The spherical hologram surface has a radii of 10cm. Wave propagation was simulated from each object surface to the hologram surface. Both the computed object wavefields are added at the hologram surface along with the reference wavefield. The intensity of this added wavefield will generate the spherical hologram which is shown in Figure 7.10.

![Figure 7.8: Object-1.](image)

![Figure 7.9: Object-2.](image)

Now, using the proposed method the reconstruction of both the objects onto their individual surfaces was attempted. As mentioned earlier the reconstruction was done by simulating wave propagation back to the object surface using the conjugate of the reference wave. The individual reconstructions of object 1 and object 2 are shown in Figure 7.11 and Figure 7.12 respectively. From Figure 7.11 it can be seen that the
reconstruction of object 1 is sharp while object 2 is blurred. Similarly Figure. 7.12 also reconstructs only object 2 with sharp focus. These reconstructions confirm that the proposed formula could demonstrate 3D reconstructions of a spherical hologram successfully.
7.4 Conclusion

The solution to Helmholtz wave equation in spherical coordinates is derived using variable separable method. The spherical wave spectrum and transfer function were defined from the solution. A formula for computing the wave propagation from irradiating spherical surfaces is devised. A fast computation method that evaluates the wave propagation formula in $O(N(\log N))^2$ operations was suggested. The proposed method was tested for correctness using simulated experiments. Generation and reconstruction of a spherical hologram for a spherical object was also successful. Three dimensional reconstructions of a spherical hologram was also demonstrated. Hence a new and fast computation method for computer generated spherical holograms is realized.
Chapter 8

Summary and Future Work

8.1 Summary of the Work

This research work is an attempt to develop spectral wave propagation formula for cylindrical and spherical systems. Then test its efficiency using computer generated cylindrical and spherical holograms. Since cylindrical and spherical holograms come under non-planar holograms, this research is categorised as a study of non-planar holograms. The whole research work can be summarized as follows:

Holograms are usually recorded on flat surfaces, which limits its ability to reconstruct the object from all sides. Among all the holographic recording and reconstruction techniques available, cylindrical holograms had the capacity of 360° reconstruction in the azimuthal direction, while spherical holograms could reconstruct for 360° in both azimuthal and polar direction. However optical setup to provide proper illumination for recording and reconstructing a cylindrical or spherical hologram is very difficult. More over Optical holographic recording demands very high stability (vibration isolation) of the object to be recorded. All these constraints can be overcome, if cylindrical and spherical holograms were generated on a computer. But very less work has been done with respect to cylindrical and spherical computer generated holography and it is still in its infancy. Scalar diffraction theories are the most suitable and the most used ones to simulate wave propagation for computer generation of holograms. Among the scalar diffraction formulae, the spectral propagation formula involves only two FFT operations and is the fastest. However such a formula has not been reported for cylindrical or spherical computer generated holography yet. Hence this work was an attempt to devise such a computational method to generate cylindrical and spherical holograms and verify the results.
Accordingly, first the spectral propagation formula in cylindrical coordinates was devised which turned out to have the same form as that of spectral propagation from Cartesian co-ordinates. The transfer function that defines the spectral wave propagation was defined which was a ratio of Hankel functions of first kind. The spectral decomposition of wavfield on the cylindrical surfaces was also defined and is known as the Helical wave spectrum. Using the developed formula a numerical procedure was designed that could use FFT for its calculations. The usage of FFT was made possible by assuming the object and hologram surfaces to be concentric cylinders and hence shift invariant. The numerical computation had similar form to angular spectrum method and can be done using only two FFT operations and simple multiplication. The generated cylindrical holograms could reconstruct the objects properly using this numerical method. Reconstructions using direct integration method also produced the same results and hence the correctness of the hologram was verified. The numerical method also was able to reconstruct multi surfaced objects back onto their corresponding object surface from a single cylindrical hologram. A 3-dimensional object that had 64 cylindrical segments was also reconstructed successfully. The whole work was done using open source software and tools alone and hence this work can be reproduced at no additional cost. Finally the applicability of the method was tested by attempting an optical reconstruction of the cylindrical hologram. Though the optical reconstructions were noisy the object could be recognized and hence the proposed method can be successfully applied to real world problems.

Second, the spectral propagation formula for spherical systems was derived as boundary value solutions to the Helmholtz wave equation. From the solutions the transfer function for wave propagation from one spherical surface to another was defined. The decomposition of wavefield on a spherical surface was also defined. From these definitions the spectral propagation formula for spherical surfaces was developed. This formula was similar to the angular spectrum formula, only difference being the usage of spherical transform operation instead of Fourier transform operation. Here again fast computation was made possible by assuming the object and hologram to be concentric spherical surfaces and also by using the theorem of band limited functions on a spherical surface.

A fast numerical method was chosen to evaluate the proposed formula. Initially the developed method was tested by attempting to reproduce the already reported results for spherical CGH. This was successful and then the method was tested to compile with the basic laws of diffraction and interference. To test the method’s applicability in holography, a spherical hologram was generated for a spherical object and successfully reconstructed back using this spectral method. Finally 3D reconstruction capabilities were also tested by assuming the object to have two spherical surfaces and computing the hologram. From the single hologram both the objects were reconstructed back to
their individual surfaces. Hence the method was successfully tested to be applicable in spherical holography.

8.2 Original Findings of the Work

The findings of this work can be stated as follows

- A new computation method has been developed and successfully tested for simulated and optical cylindrical CGH experiments. The method is based on wave propagation is spectral domain.

- The proposed computation method is more efficient than the other methods in i) Calculation speed, ii) Accuracy and iii) Sampling condition.

- A new computation method has been developed and successfully tested for simulated spherical CGH experiment. The method is based on boundary value solutions to Helmholtz equation in spherical co-ordinates.

- The proposed method is more efficient than the other methods in i) Calculation speed, ii) Accuracy and iii) Sampling condition.

Hence this research work has introduced two important formulas for Computer generated holography in non-planar surfaces with clear advantages over the already reported ones. This is represented in the table as shown below as figure 8.1.

8.3 Suggested Future Work

The formula proposed for cylindrical and spherical CGH are analogous to the angular spectrum formula for plane CGH. However the numerical evaluation of angular spectrum method is error prone when the distance of propagation increases. Hence usually the approximated Fresnel diffraction formula usually used in plane CGH when the object to hologram distance is large. Similarly the transfer functions of the proposed spectral formulas for cylindrical and spherical CGH are also error prone to large distances of propagation. However the interesting fact is that both these transfer functions have their asymptotic forms and hence an approximated formula similar to the Fresnel formula is possible. Such a formula will be more efficient and error free. It will take this method more close to realising applications. Hence this will be the immediate future work to the proposed method.

Digital holographic microscopes are becoming popular now a days and are commercially available today. But they can reconstruct the 3-dimensional profile from
### Figure 8.1: Diffraction theories for CGH (Original findings shown in red)

**Plane CGH**

**Direct Integration**

\[
H(x, y) = \int \int \frac{O(x_0, y_0) \exp(i k L)}{L} \, dx \, dy
\]

\[
L = \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}
\]

**Convolution**

\[
H(x, y) = O(x_0, y_0) \otimes PSF
\]

**Angular Spectrum**

\[
H(x, y) = \text{FFT}^{-1}[\text{FFT}[O(x_0, y_0)] \times TF]
\]

\[
TF = \exp[i k s \sqrt{1 - (\lambda f_s)^2 - (\lambda f_t)^2}]
\]

**Cylindrical CGH**

**Direct Integration**

\[
H(\theta, y) = \int \int \frac{O(\theta_0, y_0) \exp(i k L)}{L} \, d\theta \, dy
\]

\[
L = \sqrt{R^2 + r^2 - 2 R r \cos(\theta - \theta_0) + (y - y_0)^2}
\]

**Convolution**

\[
H(\theta, y) = O(\theta_0, y_0) \otimes PSF
\]

**Angular Spectrum**

\[
H(\theta, y) = \text{FFT}^{-1}[\text{FFT}[O(\theta_0, y_0)] \times TF]
\]

\[
TF = \frac{H^{(1)}(k_r)}{H^{(1)}(k_s)}
\]

**Spherical CGH**

**Direct Integration**

\[
H(\theta, \phi) = \int \int \frac{O(\theta_0, \phi_0) \exp(i k L)}{L} \, d\theta \, d\phi
\]

\[
L = \sqrt{R^2 + r^2 - 2 R r \sin(\theta) \sin(\theta_0) + \cos(\theta) \cos(\theta_0) \cos(\phi - \phi_0)}
\]

**Convolution**

\[
H(\theta, \phi) = O(\theta_0, \phi_0) \otimes PSF
\]

**Angular Spectrum**

\[
H(\theta, \phi) = SHT^{-1}[SHT[O(\theta_0, \phi_0)] \times TF]
\]

\[
TF = \frac{h^{(1)}(kr)}{h^{(1)}(k_s)}
\]
Summary and Conclusion

only one side of the object. A whole volumetric reconstruction of the entire object is not possible. However the proposed numerical method has whole volumetric reconstruction abilities as demonstrated in Chapter 4. In other words, this would be a technique very similar to Magnetic resonance imaging and Computed tomography except for the fact that, only surface morphology is reconstructed when using a hologram. Now the most challenging task will be to make an optical setup to illuminate the whole object coherently from all sides and also to generate a corresponding reference. If this is successfully achieved then the interference pattern on the cylindrical surface could be recorded by placing a line scan camera on a rotational stage. Then the proposed numerical method can be used to reconstruct the image back in the computer. Since this is a near field diffraction formula, details of the order of few micrometers can be resolved. Lateral resolution can be further improved upto the diffraction limit by using pixel sub sampling. Axial resolution can be improved to sub wavelength dimensions using phase shifting technique. However construction of an optical setup to incorporate all these improvements is challenging. Hence a digital cylindrical holographic microscope that could reveal microscopic surface details of 3-dimensional objects in 360° can be realized.

In the case of spherical CGH a general expression for the sampling condition of the transfer function has not been derived yet. This is very important for realising applications and will be the immediate future work to the proposed method. Farfield diffraction patterns from spherical surfaces of similar problems (in acoustics and geophysics) have shown that far field diffraction pattern is directional and not uniformly distributed over a spherical surface. Hence we do not need a spherical detector (which is not available now) or scan over the whole spherical surface in order to holographically image a spherical volume. This interesting fact takes it much closer to realizing practical applications. The availability of optic fibers will also be a great aid in this process. When considering holographic display, printing a spherical hologram on a flexible film and reconstructing using a laser will be a challenging task. However with the availability of high precision lithographic machines and optic fibers this is not an impossible task.

Finally it is also worth referring to Horvath [95] for more interesting facts on the importance of this research work to the future of lasers. He quotes that “For the past half century, both scientists and the public have come to think of lasers as producing line-like beams. Yet there is nothing about the laser that requires light emission to occur in a single dimension. Though initially forgotten and ignored, multidimensional lasers may define the next era of technologies evolution”. When such a situation happens, this research work will be widely put to use in developing applications.
8.4 Conclusion

The two main bottle necks in the field of computer generated holography and digital holography are i) computation speed and ii) resolution of dynamic recording and display devices. This, along with the unavailability of cylindrical and spherical shaped recording and display devices, is the main constraint in realizing the proposed method. But with the sudden development in nanotechnology, we are not far from the days to have recording and display devices with pixel pitch in the order of hundreds of nanometers. It may also be possible to produce them in cylindrical or spherical shape. Then the other constraint, which is computation speed, can be improved by using the proposed method, which is the fastest available. Hence we believe, this work which is a new and fast calculation method will be a small step towards realizing recording and reconstruction of computer generated cylindrical and spherical holograms.
Bibliography


